ELEMENTARY GEOMETRY

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PREFACE

A REPORT on the teaching of elementary geometry was drawn up by a special committee appointed by the I.A.A.M. and issued in 1923. This report contains some valuable practical suggestions and a detailed sequence of theorems with notes on procedure. The present text-book incorporates all the recommendations made in that report and in particular adopts the sequence therein scheduled, both for theorems and constructions.

The treatment falls into three stages, the last of which occupies most of the book. The first stage deals with the meaning of the fundamental concepts and the second with the construction of a broad axiomatic basis. The time that can profitably be allotted to these two stages naturally depends on local conditions. In some schools it is accompanied by practical outdoor work and wood-work. But whether the time given to these two stages is a single term, or extends to a year's course, these stages remain distinctive and form the foundation on which the final stage, the deductive development, rests.

In the final stage, proofs of those theorems which the report considers should not be set in any public examination have been omitted from the main text, but will be found in an Appendix because they are still required by some Examining Boards. A second appendix contains notes on limit methods of proof, and a third appendix gives a few theorems of a more advanced character than those included in the report's schedule.

A report on the teaching of geometry has also been issued recently by the Mathematical Association. This document advocates two fundamental changes of method, but it also contains a number of practical suggestions which the author has found helpful, especially in connection with the treatment of loci.

It is hoped that the collection of examples and riders will be considered the most useful part of the book. The author has been collecting and inventing riders for the last twenty years, because in his view rider-work (which includes numerical applications) is the chief source of interest in geometry and is the best test of geometrical progress.

C. V. D.

February 1925.

NOTE

The author has prepared for the use of teachers a book containing solutions of the riders, hints for solution of numerical exercises, statements of constructions in all cases which are not obvious applications of the bookwork, and those answers which it was considered advisable to omit from the table of Answers at the end of the book. Price 3s. 6d. net.

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SYMBOLS

$\therefore$	therefore.	$<$	less than
$=$	equivalent.	$\angle$	angle.
$\equiv$	congruent.	rt. $\angle$	right angle.
$\backsimeq$	approximately equal.	$\triangle$	triangle.
$\sim$	the difference between X and Y is represented by $X \sim Y$.	$\parallel$	parallel.
		$\parallel$gram	parallelogram.
$>$	greater than.	$\bigcirc$ce	circumference.

PART I.

STAGE A.

FUNDAMENTAL CONCEPTS.

Solid.

Any object which has at any moment a definite shape is called a **geometrical solid**, *e.g.* a match-box, a tennis net or a sponge. Most solids are irregular in shape. There are some simple forms such as a cube, cuboid, sphere, cylinder, prism, cone, pyramid which must be recognised by name, when seen. Models of these should be available and some of them should be constructed, in thin cardboard.

Surface.

A solid is bounded by **surfaces**. A surface has no thickness : the surface marks the separation between the outside space and the space occupied by the solid.

A surface is either **plane** or **curved**, *e.g.* the surface of a ball is curved, the surface of a table is plane or as nearly plane as the carpenter can make it.

The **planeness** of a surface is tested (*e.g.* by a carpenter) by using a straight edge. The edge is applied to the surface ; if it fits it for the whole of its length *and in all positions*, the surface is plane.

Line.

The intersection of two surfaces determines a line. A line has neither thickness nor breadth. It may also be regarded as marking the division of a surface into two parts, *e.g.* suppose part of a wall is painted red and the remainder black, the

separation between the red and the black portions is a line (which is neither red nor black). A line is either **straight** or **curved**, *e.g.* two plane walls of a house intersect in a straight line, the seam on the cover of a tennis ball is a curved line.

Straight lines are drawn by using a straight edge or ruler. The **straightness** of a ruler is tested as follows: make two pinpricks in the paper and join them by a line, using the ruler. then reverse the direction of the ruler and draw a line joining them again. If these two lines coincide, the ruler gives a straight edge.

Point.

The intersection of two lines determines a **point**. A point has neither thickness nor breadth nor length; it may be regarded as marking a position in space, *e.g.* two edges of a box meet at a point which marks the position of a corner of the box.

Representation of a Point.

The right way. **The wrong way.**

FIG. 1.

Never represent a point by a blob, but always represent it by two lines cutting each other.

EXERCISE I.

1. Name three objects which are approximately spheres.

2. Is it possible to draw a straight line on the surface of (i) a sphere, (ii) a cylinder, (iii) a cone?

3. Name two objects which are such that all their surfaces are plane.

4. What are the surfaces which bound a worn penny?

5. Is it possible to *draw* a straight line at all?

6. Is the film of a soap bubble a surface or is it a solid?

7. Name two objects each of which is bounded partly by plane surfaces and partly by curved surfaces.

8. Is a sheet of paper a solid or a surface ?

9. Is the coat of paint on a wall a solid or a surface ?

10. Name an object shaped like a cylinder.

11. How can you fold a flat sheet of paper to form the curved surface of a cylinder ? What is the original shape of the paper ? Does any straight line ruled on the flat sheet remain straight when the sheet is folded ?

12. Name an object shaped like a cone.

13. How can you fold a flat sheet of paper to form the curved surface of a cone ? What is the original shape of the paper ? Does any straight line ruled on the flat sheet remain straight when the sheet is folded ?

14. What kind of surface is the curved surface of (i) a bucket, (ii) a water pipe, (iii) a concave mirror ?

15. Fill up the following table :

	Number of Faces.	Number of Corners.	Number of Edges.	
	F	C	E	F+C−E
Cube - - - -				
Cuboid - - -				
Triangular prism - -				
Triangular pyramid -				
5-sided prism - -				
4-sided pyramid - -				
A solid L - - -				

What conclusion can you draw from this table ? Check by considering the solid obtained by cutting the head off a 4-sided pyramid.

16. A circular wooden cylinder floats in water; what is the shape of the section made by the water-surface if the axis of the cylinder is (i) vertical, (ii) horizontal ?

17. A circular wooden cone floats in water, with its axis vertical, what is the shape of the water-line ? What would be the shape if the cone is held half submerged and the axis horizontal ?

18. How many faces of a cube is it possible to see at the same time ?

19. Draw a sketch of a cube placed so that (i) 2 faces, (ii) 3 faces can be seen.

20. Draw a sketch of (i) a circular cylinder, (ii) a triangular prism, (iii) a triangular pyramid.

21. What measurements would you make to fix the size of (i) an ordinary wooden match-box, (ii) a postcard, (iii) a curtain pole, (iv) a curtain ring ?

22. Cut out on thin cardboard a figure shaped as in Fig. 2 Cut along the dotted lines, so that the cardboard can be folded about these lines. Then fold the cardboard so as to make a cardboard box. Which lines in the original figure must be equal ? *This figure is called the net of the box.*

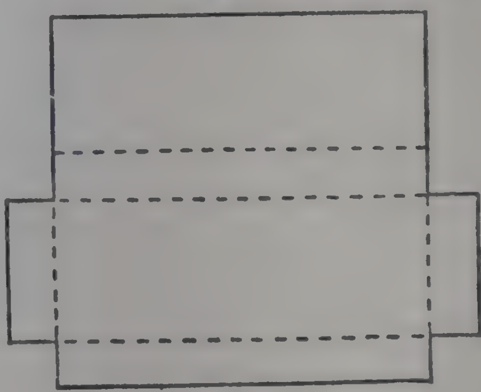


FIG. 2.

23. What is the net of a box without a lid ? Indicate which lines in the net must be of equal length ?

24. What is the net of a closed cubical box ?

25. Fig. 3 represents the net of a triangular prism. Which lines in the figure must be equal ? Draw the net and cut it out and fold it to form a prism.

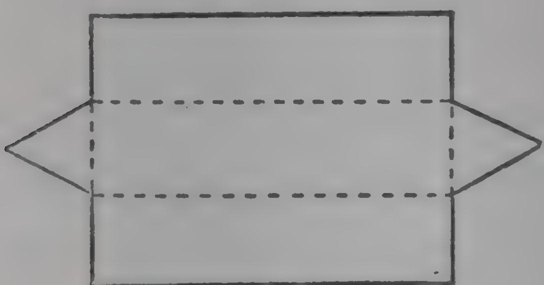


FIG. 3.

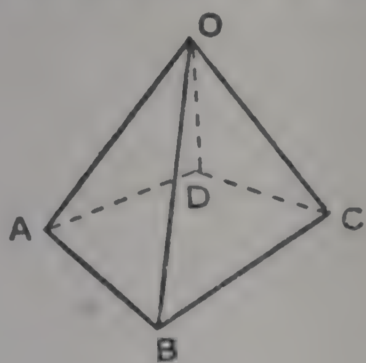


FIG. 4.

26. Fig. 4 represents a pyramid on a square base ABCD with its vertex at O ; it is made of thin cardboard. The edges OA, OB, OC, OD are cut and the surface is then folded out flat. What is

the shape of the net, so obtained? Draw it and indicate which lines in the figure *must* be of equal length.

27. Draw a sketch of a pyramid standing on a triangular base. Draw the net obtained by cutting down three of the edges which meet at one corner. Suppose the edges of the pyramid are of unequal length, indicate what lines in the net *must* be equal in length.

28. A circular cylinder is made of thin paper and has both ends closed. Draw the net from which it could be constructed. What connection is there between the lengths of different parts of the net?

29. What solid could be constructed from the net given in Fig. 5? Is it necessary that (i) AQ=AR, (ii) AR=RB, (iii) AB=AC?

30. Use Fig. 5 to construct in thin cardboard or stiff paper, a pyramid on a triangular base, (i) with all edges equal, (ii) with its edges unequal.

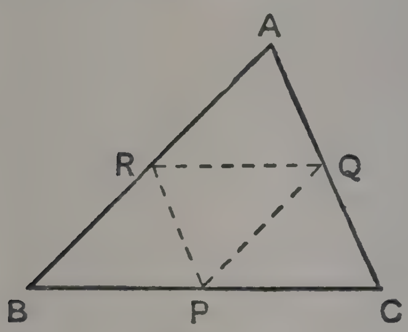


FIG. 5.

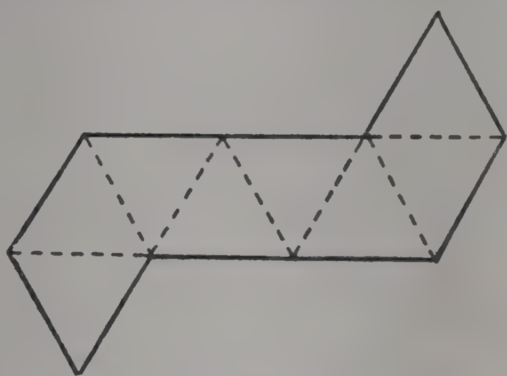


FIG. 6.

31. Use the net in Fig. 6 to construct a regular octahedron (*i.e.* a solid bounded by 8 faces each of which is an equilateral triangle).

Measurement of Straight Lines.

Two scales are marked on the ruler, one shows inches and tenths of a inch, the other shows centimetres (cm.) and tenths

FIG. 7.

of a centimetre, or millimetres (mm.). Measurements of length should always be given up in decimals.

Measure the line AB in inches.

Place the zero graduation opposite A and the edge of the ruler along AB, and note what point of the scale is opposite B. This is best done by standing the ruler on its edge instead of putting it flat on the paper.

We see that AB lies between 1·4 in. and 1·5 in., and we then judge by eye the number of hundredths of an inch, and so estimate AB as equal to 1·43 in.

Another method of measurement consists in taking the dividers and opening them out so that one point rests on A and the other point on B, and then reading off the answer by placing the dividers on the graduated ruler scale.

When drawing a line of given length, start by drawing a line which is too long, mark a point close to one end by drawing a short line across (as in Fig. 7 at A), and then cut off the required length with your compasses. Fig. 7 shows the correct way of representing a finite straight line, *i.e.* a line of definite length : its extremities are marked by short crossing lines, *not* by blobs on the line.

EXERCISE II.

1. Measure as accurately as you can in inches *and* centimetres the lengths of the lines in Fig. 8, and fill in the following table :

	AB	BC	CA	DE	EF	FD	KL	LM	MK
Inches - -									
Centimetres -									

2. Measure as accurately as you can in ins. and cms. the distances between the following pairs of points in Fig. 8:

(i) A, K ; (ii) K, C ; (iii) B, L ; (iv) A, M ; (v) F, K.

3. Draw a straight line and cut off from it a length of 5″ ; measure it in cm., and so find the number of cm. in 1 inch.

4. Draw a line and cut off from it a length of 10 cm. ; measure it in inches and hence express 1 cm. in inches.

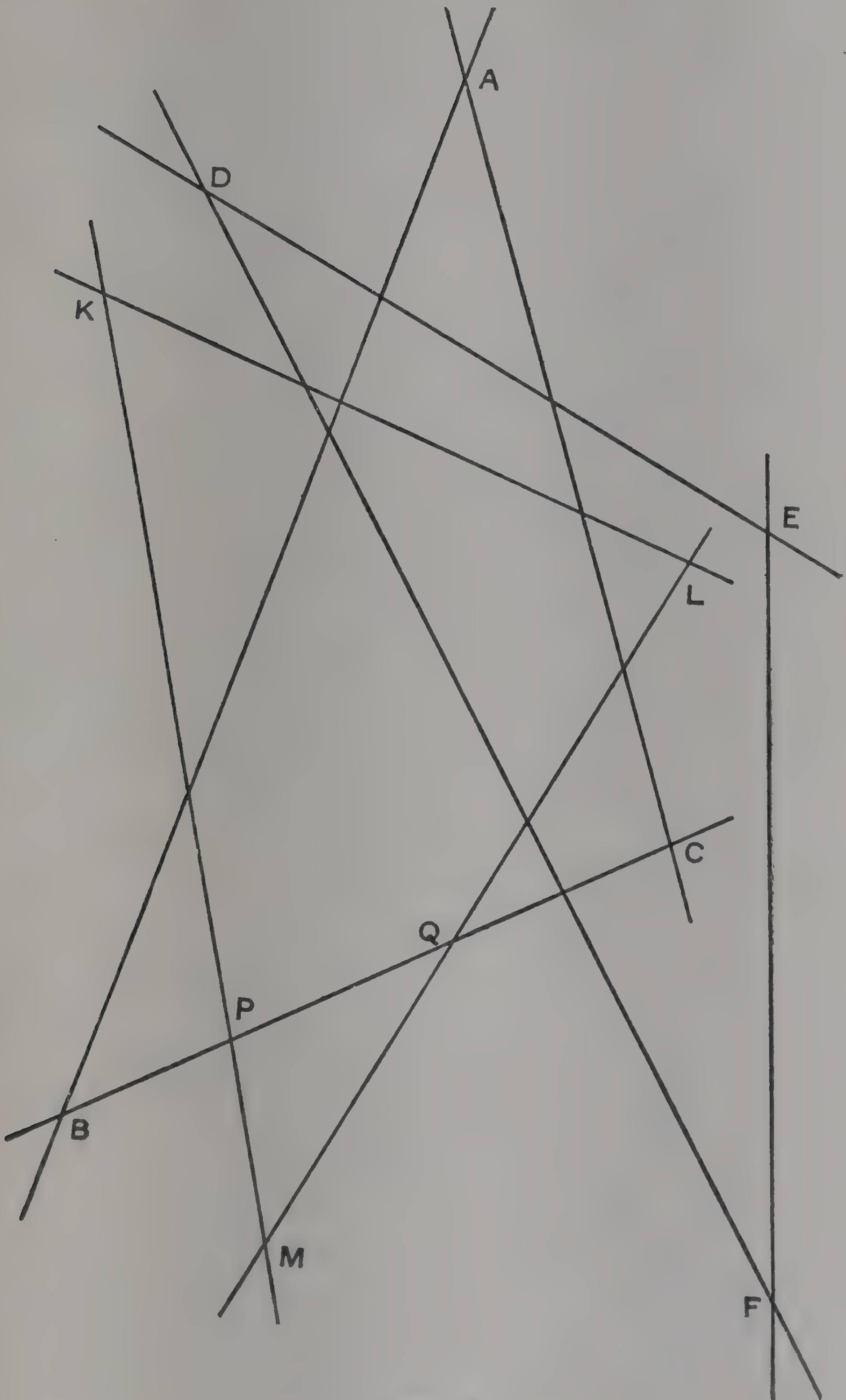


FIG. 8.

5. Draw a straight line across your sheet of paper and mark off by eye lengths of 4 cm., 7 cm., 2 in.; then measure them and write down your errors (+ or −).

6. In Fig. 8 measure in cm. the lengths of BQ, PC, BC, PQ. What are the values of (i) BQ + PC, (ii) BC + PQ ?

7. Measure in inches and cm. the length of this page. Taking $1'' = 2{\cdot}54$ cm. approx., find how far your measurements agree with each other.

8. In Fig. 8, guess the lengths of (i) AL in inches, (ii) CE in cm., and check by measurement.

9. Find by measurement which points in Fig. 8 are more than twice as far from A as L is.

10. Draw a straight line across your paper; mark the middle point of the line by eye and measure the two parts. How far is the point you have marked from the real mid-point of the line ?

11. Draw a straight line across your paper and divide it by eye into three equal parts. Measure the three parts.

12. Draw a straight line across your paper and use your dividers (i) to bisect it, (ii) to trisect it, by trial. Check by measurement.

13. Draw two lines AOB, COD crossing each other, and such that $AO = OB = 1{\cdot}3''$, $CO = OD = 2{\cdot}1''$. Join AC, AD, BC, BD, and measure these four lines in inches. What do you notice about them.

14. The hands of a clock are 7″ and 4″ long. What is (i) the greatest distance, (ii) the least distance between the tips of the two hands during the day ?

Direction. Horizontal and Vertical Lines and Planes.

EXERCISE III. (Oral).

1. Point (i) vertically downwards, (ii) vertically upwards.

2. Name some vertical lines in the room.

3. How can you test experimentally whether two points are in the same vertical line ?

4. What is a plumb line ?

5. Do two vertical lines meet, if so at what point ?

6. Point in a horizontal direction. Point northwards; are you pointing horizontally ? Point southwards, eastwards, westwards, are these horizontal directions ?

7. Name some horizontal lines in the room.

8. How can you test experimentally whether a line is horizontal ?

9. What is a spirit level ?

10. Place a cuboid so that one of its edges is vertical. State how many of its edges are vertical and how many are horizontal.

11. Can you hold a cuboid so that one of its edges is horizontal but none of its edges are vertical ?

A plane which passes through the centre of the earth is called a **vertical plane**.

A plane which passes through *two* intersecting horizontal straight lines is called a **horizontal plane.**

12. Name some vertical planes in the room.

13. Name some horizontal planes in the room.

14. How does a builder find out whether the face of a wall is a vertical plane ?

15. When a billiard table is set up, how does the workman find out whether the surface is a horizontal plane ?

16. Is a smooth surface necessarily a level surface ?

17. A plane contains a vertical line, must it be a vertical plane ?

18. A plane contains a horizontal line, must it be a horizontal plane ?

19. A plane contains 2 horizontal lines, must it be a horizontal plane ? Keep your book on the table and hold the cover a little open so that the surface of the cover is not a horizontal plane, can you draw several horizontal lines on it ?

20. Can you draw a horizontal line on a vertical plane ?

21. Can you draw a vertical line on a horizontal plane ?

22. Name two vertical planes in the room which cut each other. In what sort of a line do they intersect ?

23. How many (i) vertical lines, (ii) horizontal lines can you draw through one given point ?

24. How many vertical planes can you draw through (i) a given vertical line, (ii) a given line which is not vertical ?

25. How many horizontal planes can you draw through (i) a given horizontal line, (ii) a given line which is not horizontal ?

26. How many (i) horizontal lines, (ii) vertical lines can you draw on an oblique plane, *i.e.* a plane which is neither horizontal nor vertical ?

27. What is the difference in the tests for (i) a plane surface, (ii) a level surface ?

28. A cuboid is held so that one of its surfaces is a vertical plane. How many of its other faces *must* be (i) vertical planes, (ii) horizontal planes ? How many of its edges *must* be (i) vertical lines, (ii) horizontal lines ?

29. A cuboid is held so that one of its surfaces is a horizontal plane. How many of its other faces *must* be (i) horizontal planes, (ii) vertical planes ? How many of its edges must be (i) horizontal lines, (ii) vertical lines ?

30. Is it possible for two oblique planes to intersect in (i) a horizontal line, (ii) a vertical line ?

Lines in the same direction.

EXERCISE IV. (Oral).

1. Two people both point eastwards. Are they pointing in the same direction ?

2. Two people both point horizontally. Must they both be pointing in the same direction ?

3. Name several lines in the room which have the same direction.

4. Name several vertical lines in the room ; are these in the same direction ?

5. Name several horizontal lines in the room which have different directions.

6. Name several oblique lines in the room which have the same direction.

7. Hold a cuboid so that its edges are oblique lines, and name those which have the same direction.

8. If two people point to the same star, are they pointing in the same direction ?

9. If two people point to the same object, are they necessarily pointing in the same direction ?

10. Has a vertical line in London the same direction as a vertical line in New York ?

11. Place two set-squares side by side, flat on the paper and touching each other. Hold one still and slide the other along it. How does this give lines with the same direction ?

12. Do the shadows of two telegraph poles in the sunlight give lines in the same direction ?

13. How could you test whether two lines drawn on a sheet of paper are in the same direction ?

14. *Any* line is drawn on the floor of a room. Is it *always* possible to draw (i) on the ceiling, (ii) on a wall of the room a line in the same direction ?

Angle.

In one hour the minute hand of a clock makes one complete turn. In a quarter of an hour the amount of turning is one quarter of a complete turn. This amount of turning is called a **right angle**, so that four right angles is the same as one complete turn. If you stand at A, looking along the line AB, and then turn so as to look along the line AC, you are said to have turned through an angle, and the size of this angle is the amount of turning from one position to the other.

EXERCISE V. (Oral).

(Give your answer in right angles and fractions of a right angle.)

What angle is turned through in the following ?

1. Face North, now turn and face East.
2. Face East, now turn and face West.
3. Face North, now turn the way of the sun and face West.
4. Face West, now turn and face South.
5. Face North-east, turn and face South-east.
6. Face N.-W., turn and face S.-E.
7. Face S.-W., turn and face N.
8. Face E., turn and face S.-E.
9. Face W., turn and face S.-E.
10. Face E., turn round twice completely.
11. The minute hand in 30 minutes, 45 minutes.

12. The hour-hand in 3 hours, 12 hours, 24 hours.

13. The minute hand is $2\frac{1}{2}$ hours.

14. The seconds hand in 5 minutes, 7 minutes.

15. At the order right turn, left turn, about turn, half left turn.

What is the final direction in the following ?

16. Face E. and turn right-handed through 1 right angle, $1\frac{1}{2}$ right angles.

17. Face W. and turn left-handed through 1 right angle, $\frac{1}{2}$ right angle, $2\frac{1}{2}$ right angles.

18. Face S. and turn right-handed through 3 right angles, $\frac{1}{2}$ right angle.

19. The minute hand starting at 3.20 p.m. and turning through 2 right angles, $\frac{2}{3}$ right angle.

20. The minute hand starting at 3.50 p.m. and turning through 1 right angle, $2\frac{1}{3}$ right angles.

21. Estimate by eye the size of the angles in Fig. 9.

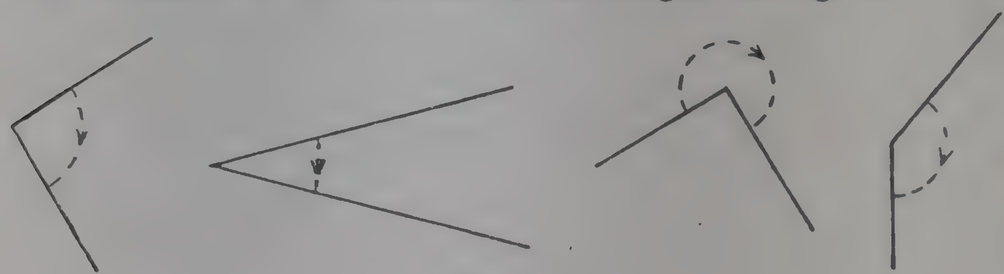


FIG. 9.

22. Sketch free-hand angles of the following sizes : $\frac{1}{4}$, $2\frac{1}{2}$, $1\frac{1}{4}$, 4, 3 right angles.

23. Through what angle does the spoke of a wheel turn when the wheel makes 50 revolutions ?

24. The direction in which a cricket ball leaves the bowler's hand is altered by the batsman by one-third of a right angle towards the off. What fieldsman does it go to ?

25. A man pushes a marking-out machine along the outside lines of a tennis court, starting at the middle of one side. What angle does he turn through at each corner ? What is the total angle he has turned through when he gets back to the starting point ?

26. Draw a rough figure representing three places A, B, C if B is due south of A and C is South-west of B. A man walks from

A to B and then from B to C. Mark on your figure the *angle through which he turns* at B. What is its size ?

27. A man is walking due East along a road, he then takes a side road running South-east and a little later turns into another road running due East. Draw a rough figure representing his walk. Mark on your figure the *angle through which he turns* at each place. What are their sizes ?

28. A man is walking due East along a road, he then takes a side road running North-west and a little later turns into another road running due East. Draw a rough figure representing his walk. Mark on your figure the angle through which he turns at each place. What are their sizes ?

29. A man is walking due West along a road, he then takes a side road running South-east and later turns into a road running due East. Draw a rough figure representing his walk. Mark on your figure the angle through which he turns at each place. What are their sizes ?

30. ABC in Fig. 8 represents an enclosure ; a man starts at the middle point of BC and walks all round it. Draw a rough figure and mark on it the angle through which he turns at each corner. What is the total angle he has turned through when he gets back to his starting point. Is there any connection between the angle he turns through at C and the angle a man standing at C and looking along CB turns through when he turns to look along CA ?

Measurement of Angles.

Take your dividers and gradually open them out.

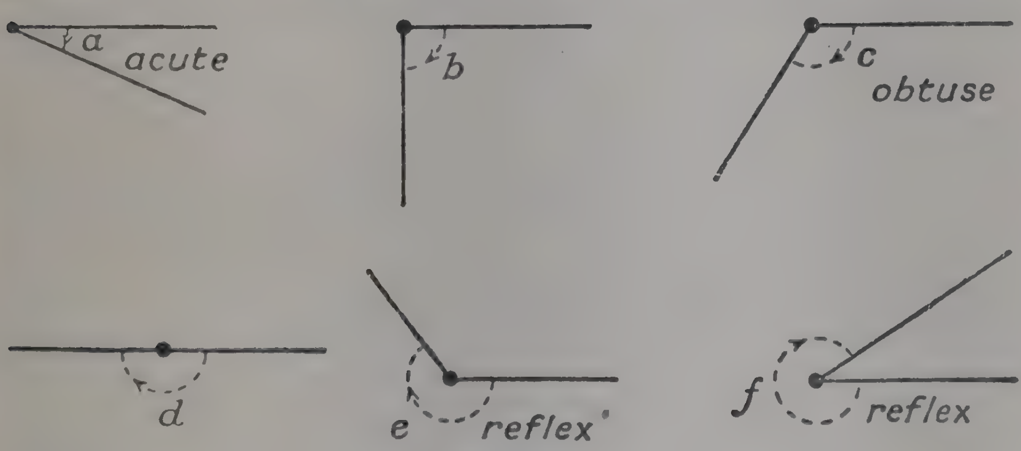


FIG. 10.

Keep one arm still and rotate the other. Fig. 10 represents successive positions.

The angle a is less than 1 right angle and is called an **acute angle.**

The angle b is **1 right angle.**

The angle c is between 1 and 2 right angles, and is called an **obtuse angle.**

The angle d is 2 right angles, and is sometimes called a **straight angle.**

The angles e, f are between 2 and 4 right angles, and are called **reflex angles**

Instead of measuring angles as fractions of a right angle, it is convenient to divide a right angle into 90 equal angles and call each of them a degree.

$$1 \text{ right angle} = 90 \text{ degrees} = 90°,$$

$$2 \text{ right angles} = 180 \text{ degrees} = 180°.$$

$$\text{Further, } 1 \text{ degree} = 60 \text{ minutes} = 60'$$

$$\text{and } 1 \text{ minute} = 60 \text{ seconds} = 60''.$$

The size of an angle is measured in degrees by using a **protractor.** Notice on your protractor that the divisions are marked with two numbers, one bigger than 90° and one smaller than 90°. You must use your common sense to decide which of these readings you are to choose in any given case.

EXERCISE VI.

1. Take your dividers and open them to an angle of about 20°; next open them a further angle of 40°. What is the total angle to which they are now opened?

2. Estimate by eye in degrees the three angles of the triangles ABC in Fig. 8, page 7. Then measure them with your protractor.

3. Express in degrees: $\frac{1}{2}$ right angle, $1\frac{1}{2}$ right angles, 4 right angles, $\frac{2}{3}$ right angle, $\frac{1}{4}$ right angle.

4. Express in right angles and fractions of a right angle: 270°, 45°, 135°, 120°, 30°, 75°.

5. Measure the angles of your set square.

6. Use your ruler to draw by eye angles of 90°, 30°, 150°, 200°. Measure these angles with your protractor, and write down your errors.

7. Use your protractor to draw angles of (i) 30°, (ii) 90°, (iii) 48°, (iv) 124°, (v) 220°, (vi) 300°; make them point different ways.

8. Measure the angles at which the following pairs of lines cut each other in Fig. 8, page 7, giving two distinct answers in each case: (i) LM, BC; (ii) FD, BC; (iii) AC, LM; (iv) DE, AB; (v) KL, AC; (vi) DE, FE.

9. Arrange, without measuring, the angles *a*, *b*, *c*, *d*, *e* in ascending order of magnitude, and state whether each angle is acute, obtuse or reflex.

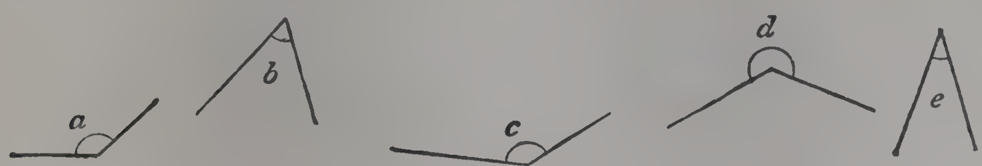


FIG. 11.

10. Draw (i) a small angle with long arms, (ii) a large angle with short arms, and estimate their sizes.

11. Is it necessary to use your protractor to find the sum of the angles *a*, *b* in Fig. 12? What is the sum?

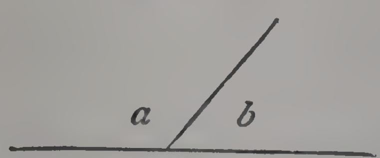


FIG. 12.

12. Is it necessary to use your protractor to find the sum of the angles *a*, *b*, *c*, *d* in Fig. 13? What is the sum?

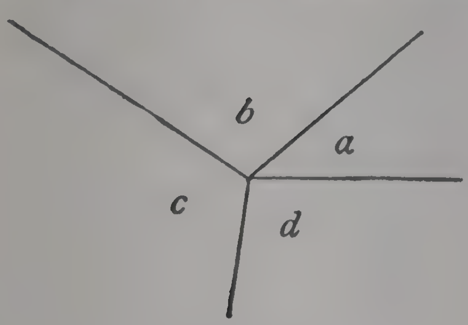


FIG. 13.

13. The hands of a clock register 2.10 ; the minute hand is then turned through 50°, and then a further 27° in the same direction. What is the total angle turned through ? What further angle must it be turned through to register (i) 2.40, (ii) 3.10 ?

14. Enlarge Fig. 14, making AB=8 cm., AD=BC=2cm., ∠DAB =90°=∠CBA. O is the mid-point of AB. Mark points on CD, such that the lines joining them to O make with OB angles of 30°, 50°,

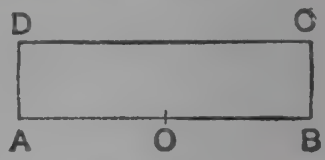


FIG. 14.

70°, 90°, 110°, 130°, 150°, and label them. Mark also points on BC, AD to indicate angles of 10°, 20°, 160°, 170°. You have now constructed a protractor of oblong shape.

STAGE B.

FUNDAMENTAL FACTS.

Angles at a Point.

EXERCISE VII. (Oral).

1. Face West, looking along OW, now turn through any angle short of an "about turn," looking along OX, now continue turning until you face East, looking along OE, so that you have made altogether an "about turn."

(i) What is the total angle turned through ?
(ii) What are the separate angles turned through ?
(iii) Is OW in the same straight line with OE ?
(iv) What can you say about the two angles the line OX makes with the straight line WOE ?

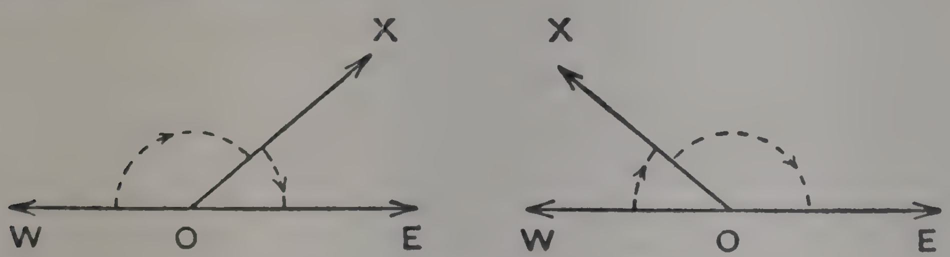


FIG. 15.

2. The time indicated by a clock is ten minutes past nine; when you next look at the clock the time is 9.33, and when you next look it is 9.40. Draw a figure showing roughly the three positions of the minute hand, and mark the two angles it has turned through. Is its first position in the same straight line as the last position ? What is the total angle it has turned through ? What can you say about the two separate angles turned through ?

3. The time indicated by a clock is five minutes past ten; when you next look at it the minute hand has turned through 130°. How many more degrees must it turn through to make half a complete turn from the start ? What can you say about the initial and final positions of the minute hand ?

4. Take your dividers and open them so that the angle between the arms is some acute angle, then open them through a further angle, so that the two arms are in one straight line. What can you say about these two angles?

5. Why are there two readings on a protractor opposite any one division? What angles do they measure? What do you notice about the two readings? If one reading is 115°, what is the other?

6. Take a pair of scissors and open them out, so that the angle between the blades is 40°. What is the angle between the handles?

7. Hold your pencil AB at its middle point O, and turn it through some angle. Does OA turn through the same angle as OB? Does it make any difference if O is not the middle point.

8. Draw two straight lines AOB, COD crossing each other at O. Place your pencil along AB and hold it fast at O. How much turning is required to make OA coincide with OC? How much turning is required to make OB coincide with OD?

The following names are important.

If two angles add up to 90° or 1 right angle, they are called **complementary**; *e.g.* 20° and 70° are complementary angles.

If two angles add up to 180° or 2 right angles, they are called **supplementary**; *e.g.* 20° and 160° are supplementary angles.

The opposite angles made by two lines crossing each other are called **vertically opposite angles**; *e.g.* in Fig. 18 x and y are vertically opposite, so are α and β.

If two angles have one arm in common and lie on opposite sides of that arm, they are called **adjacent** angles.

In Fig. 16 x and y are adjacent angles.

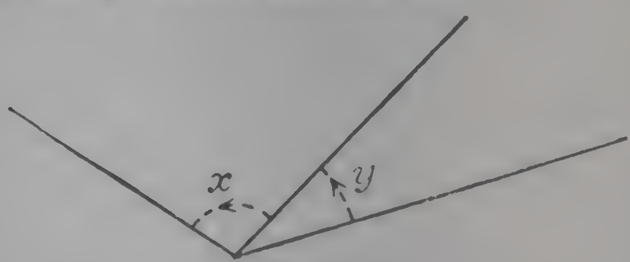


FIG. 16.

The results of these examples can be expressed in the following statements or theorems.

THEOREM 1.

(i) If a straight line CE cuts a straight line ACB at C, then $\angle ACE + \angle BCE = 180°$.

(ii) If lines CA, CB are drawn on opposite sides of a line CE such that $\angle ACE + \angle BCE = 180°$, then ACB is a straight line.

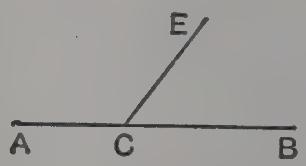


FIG. 17.

THEOREM 2.

If two straight lines intersect, the vertically opposite angles are equal.

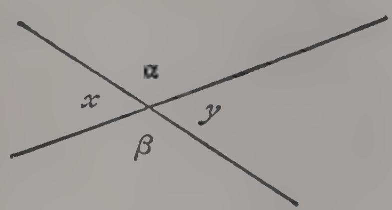


FIG. 18.

$$x = y \text{ and } \alpha = \beta.$$

EXERCISE VIII.

1. What are the supplements of 20°, 150°, 27° 45′, 92° 10′ ?

2. What are the complements of 75°, 30° 30′, 10° 48′ ?

3. A wheel has six spokes, what is the angle between two adjacent spokes ?

4. Guess the sizes of the following angles :

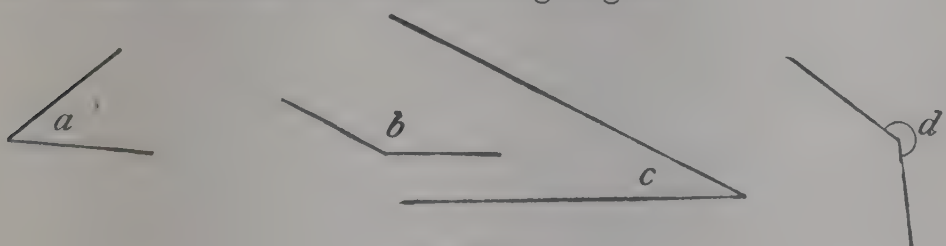


FIG. 19.

5. What is the least number of times you must turn through 17° in order to turn through (i) an obtuse angle, (ii) a reflex angle, (iii) more than one complete revolution ?

6. What is the angle between N.E. and S.E. ?

7. What is the angle between S.S.W. and E.N.E. ?

8. What is the angle between (i) 12° N. of W. and 5° E. of N. ; (ii) S.W. and E.S.E. ; (iii) 22° S. of W. and 9° N. of E. ?

9. Through what angle does the minute hand of a clock turn in 15 minutes, 5 minutes, 20 minutes, 50 minutes, 2 hours 45 minutes ?

10. Through what angle does the hour hand of a clock turn in 40 minutes, 1 hour ?

11. Through what angle has the hour hand of a clock turned, when the minute hand has turned through 30° ?

12. What is the angle between the hands of a clock at (i) 4 o'clock, (ii) ten minutes past four ?

13. A wheel makes 20 revolutions a minute, through what angle does a spoke turn each second ?

14. What equation connects x and y if x° and y° are (i) complementary, (ii) supplementary ?

15. The line OA cuts the line BOC at O ; if $\angle AOB = 2\angle AOC$, calculate $\angle AOB$.

16. What angle is equal to four times its complement ?

17. A man watching a revolving searchlight notes that he is in the dark four times as long as in the light, what angle of country does the searchlight cover at any moment ?

18. The weight in a pendulum clock falls 4 feet in 8 days ; through what angle does the hour hand turn when the weight falls 1 inch ?

19. What is the reflex angle between the directions S.W. and N.N.W. ?

20. If the earth makes one complete revolution every 24 hours, through what angle does it turn in 20 minutes ?

21. The longitude of Boston is 71° W., and of Bombay is 73° E., what is their difference of longitude ?

22. The latitude of Sydney is 33° S., and of New York is 41° N., what is their difference of latitude ?

23. Cape Town has latitude $33^\circ\ 40'$ S. and longitude $18^\circ\ 30'$ E., Cologne has latitude $50^\circ\ 55'$ N. and longitude 7° E., what is their difference of latitude and longitude ?

24. $\angle POQ = 2x°$, $\angle QOR = 3x°$, $\angle POR = 4x°$; find x.

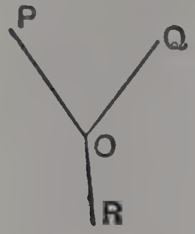


FIG. 20.

25. OP, OQ, OR, OS are 4 lines in order such that $\angle POQ = 68°$, $\angle QOR = 53°$, $\angle ROS = 129°$; find $\angle SOP$. Find also the angle between the lines bisecting $\angle POS$, $\angle QOR$.

26. OA, OB, OC are 3 lines in order such that $\angle AOB = 54°$, $\angle BOC = 24°$; OP bisects $\angle AOC$; find $\angle POB$.

27. CD is perpendicular to ACB; CE is drawn so that $\angle DCE = 23°$; find the difference between $\angle ACE$ and $\angle BCE$. What is their sum?

28. Given $\angle AOD = 145°$, $\angle BOC = 77°$, and $\angle AOB = \angle COD$; calculate $\angle AOC$ (Fig. 21).

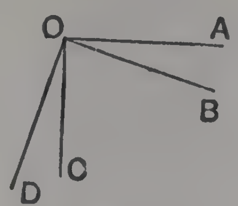


FIG. 21.

29. OA, OB, OC, OD, OE, OF are 6 lines in order such that $\angle AOB = 43°$, $\angle BOC = 67°$, $\angle COD = 70°$, $\angle DOE = 59°$, $\angle EOF = 51°$; prove that AOD and COF are straight lines. Calculate the angle between the lines bisecting $\angle AOF$ and $\angle BOC$.

30. $\angle AOB = 38°$; AO is produced to C; OP bisects $\angle BOC$; calculate reflex angle AOP.

31. OA, OB, OC, OD are 4 lines in order such that $\angle AOC = 90° = \angle BOD$; if $\angle BOC = x°$, calculate $\angle AOD$.

32. Two lines, AOB, COD intersect at O; OP bisects $\angle AOC$; if $\angle BOC = x°$, calculate $\angle DOP$.

33. OA, OC make acute angles with OB on opposite sides; OP bisects $\angle BOC$; prove $\angle AOB + \angle AOC = 2\angle AOP$.

34. The line OA cuts the line BOC at O; OP bisects $\angle AOB$; OQ bisects $\angle AOC$; prove $\angle POQ = 90°$.

35. OA, OB, OC, OD are 4 lines in order such that $\angle AOB = \angle COD$ and $\angle BOC = \angle AOD$; prove that AOC is a straight line.

Straight Lines in the Same Direction.

EXERCISE IX. (Oral).

1. Name several lines in the room which are in the same direction.

2. If 5 boys in the room point N.E., are they pointing in the same direction ?

3. If 5 boys in the room each point at the handle of the door, are they pointing in the same direction. What is your estimate of the angle between the two lines along which the two extreme boys are pointing ?

4. A boy walks north, then turns to his right through an angle of 40° and walks a short way and then turns and walks north again. What angle does he turn through the second time? Draw a figure showing his path and marking on it the angles through which he turns.

5. Draw a line AB on your paper the same length as your pencil and place two pencils along it, just covering it. Now turn one pencil through any angle about A clockwise, and turn the other through an equal angle about B clockwise. Will the two pencils now lie along lines in the same direction ? Represent what has happened by a figure.

6. Two railway signals attached, one below the other, to the same post point in the same direction when "at danger." They also point in the same direction when "at safety." Do they turn through equal angles in moving from "danger" to "safety"?

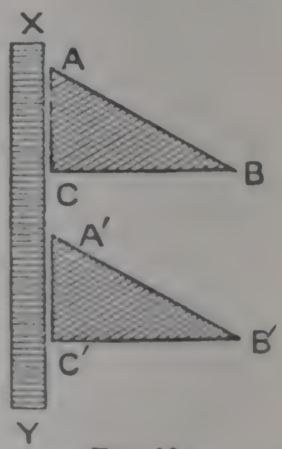


FIG. 22.

7. Two boys stand one behind the other facing in the same direction. They turn through equal angles to their right. Will they now face in the same direction ? Represent what happens by a figure.

8. Take a set square and place it against a straight edge XY (or another set square), as in Fig. 22, now slide it along so that the edge AB moves into the position A′B′. Is the direction of AB the same as the direction of A′B′ ? What can you say about the angles at which AB and A′B′ cut XY ? These angles are called corresponding angles.

Straight lines drawn in the same direction are called **parallel straight lines.**

If two straight lines are parallel, they cannot meet one another.

If two straight lines are each drawn in the same direction as a third straight line, they must have the same direction as each other: or, in other words, if two straight lines are each parallel to a third straight line, they are parallel to each other.

A straight line which cuts two or more other straight lines is called a **transversal.**

The various angles formed by a transversal with two other straight lines have special names.

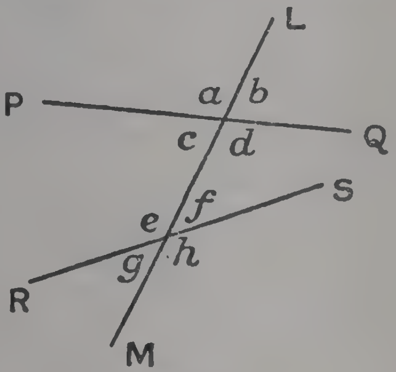


FIG. 23.

In Fig. 23 c and f are called **alternate** angles,
d and e are called **alternate** angles.

The following pairs of angles are called **corresponding**: a, e; b, f; c, g; d, h.

The angles d, f are called **interior angles** on the same side of the cutting line; so also are c, e.

EXERCISE X.

1. A man walks along the track ABCD (Fig. 24) in the direction shown, through what angles (clockwise or anti-clockwise) does he turn at B, C? Are AB and CD parallel?

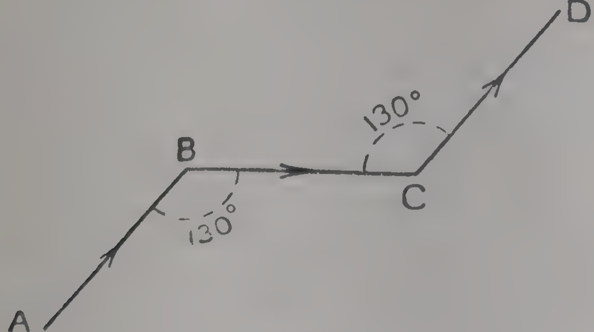


FIG. 24.

2. Put a pencil down along AB (Fig. 24), turn it anti-clockwise about B so that it lies along BC, then turn it clockwise about C so that it lies along CD. Is its new position parallel to its old position ?

3. What is the name of the marked angles in Fig. 24 ?

4. Draw Fig. 25, and draw the angle corresponding to (i) $\angle$ABC, (ii) $\angle$BCD.

5. Put a pencil down along AB (Fig. 25), turn it clockwise about B so that it lies along BC. Now turn it clockwise about C,

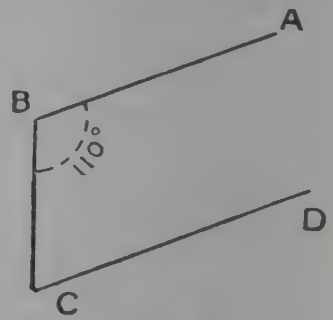


FIG. 25.

so that it has made half a complete turn altogether. If part of it lies along DC, is BA parallel to CD ? What is $\angle$BCD ?

6. Use your protractor to draw an angle ABC equal to 40°, and draw a line cutting AB so as to make two corresponding angles equal. State which these are.

7. Use your protractor to draw an angle ABC equal to 54°, and draw a line from A so as to make two alternate angles equal. State which these are.

8. If in Fig. 23, $c=72°$, $f=70°$, will PQ cut RS when produced ? If so, on which side of LM ?

9. If in Fig. 23, $b=55°$, $f=50°$, will PQ cut RS when produced ? If so, on which side of LM ?

10. If in Fig. 23, $c=72°$, and PQ is drawn in the same direction as RS, what are the sizes of g, f, h ?

11. If in Fig. 23, $d=125°$, $f=60°$, will PQ cut RS and, if so, on which side of LM ?

12. In Fig. 23, suppose M$\rightarrow$L is due North and P$\rightarrow$Q is 72° East of North, and RS is 75° East of North, will PQ meet RS when produced, and, if so, on which side of LM ?

These examples illustrate the following statements or theorems.

THEOREM 3.

In Fig. 26,

(i) If $\angle PBC = \angle BCS$, then PQ is parallel to RS.

(ii) If $\angle ABQ = \angle BCS$, then PQ is parallel to RS.

(iii) If $\angle QBC + \angle BCS = 180°$, then PQ is parallel to RS.

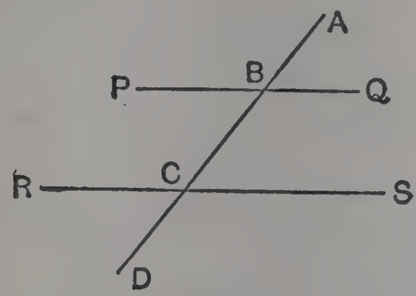


FIG. 26.

THEOREM 4.

In Fig. 26,

If PQ is parallel to RS,

Then (i) $\angle PBC = \angle BCS$ (alternate angles).

(ii) $\angle ABQ = \angle BCS$ (corresponding angles).

(iii) $\angle QBC + \angle BCS = 180°$.

THEOREM 5.

Straight lines which are parallel to the same straight line are parallel to one another.

EXERCISE XI.

1. In the following figures, a line cuts two parallel lines. What equations connect the marked angles? Give reasons.

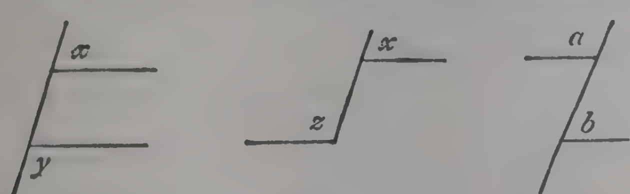


FIG. 27.

2. The following figures contain pairs of parallel lines. What equations connect the marked angles? Give reasons.

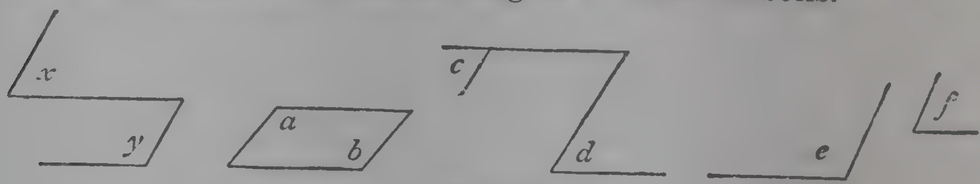


FIG. 28.

3. (i) If one angle of a parallelogram is 60°, find its other angles.

(ii) If one angle of a parallelogram is 90°, find its other angles.

4. If AB is parallel to ED, see Fig. 29, prove that

$$\angle BCD = \angle ABC + \angle CDE.$$

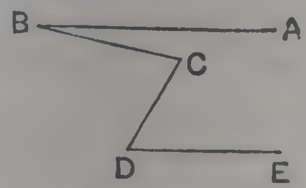


FIG. 29.

5. The side AB of the triangle ABC is produced to D; BX is drawn parallel to AC; $\angle BAC = 32°$, $\angle BCA = 47°$; find the remaining angles in the figure and the value of $\angle BAC + \angle BCA + \angle ABC$.

6. If AB is parallel to DE, see Fig. 30, prove that

$$\angle ABC + \angle CDE = 180° + \angle BCD.$$

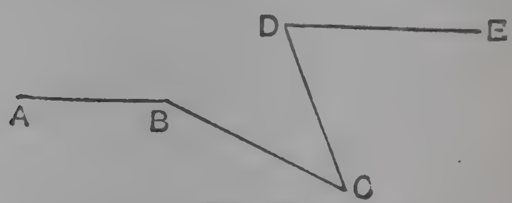


FIG. 30.

(Draw CF parallel to DE.)

7. In Fig. 31, prove that AB is parallel to ED.

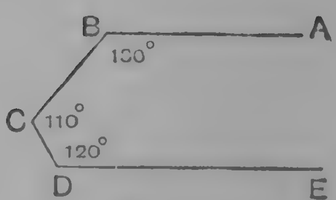


FIG. 31.

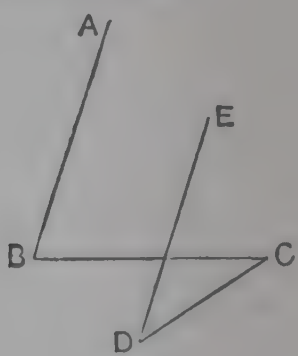


FIG. 32.

8. In Fig. 32, if $\angle ABC = 74°$, $\angle EDC = 38°$, $\angle BCD = 36°$, prove ED is parallel to AB.

9. ABCD is a quadrilateral; if AB is parallel to DC, prove that $\angle DAB - \angle DCB = \angle ABC - \angle ADC$.

10. In Fig. 33, if AB is parallel to DE, prove that $x+y-z$ equals two right angles.

11. A line AC cuts two parallel lines AB, CD; B and D are on the same side of AC; the lines bisecting the angles CAB, ACD meet at O; prove that $\angle AOC = 90°$.

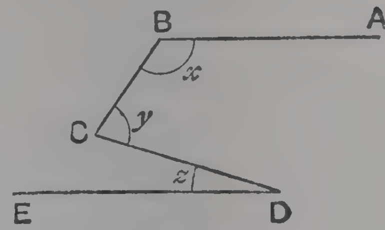


FIG. 33.

Angles of a Triangle and Polygon.

Note.—The following definitions are collected here for convenience. They should, however, be introduced incidentally as occasion requires, not all at once.

Definitions.

A plane figure bounded by three straight lines is called a **triangle**.

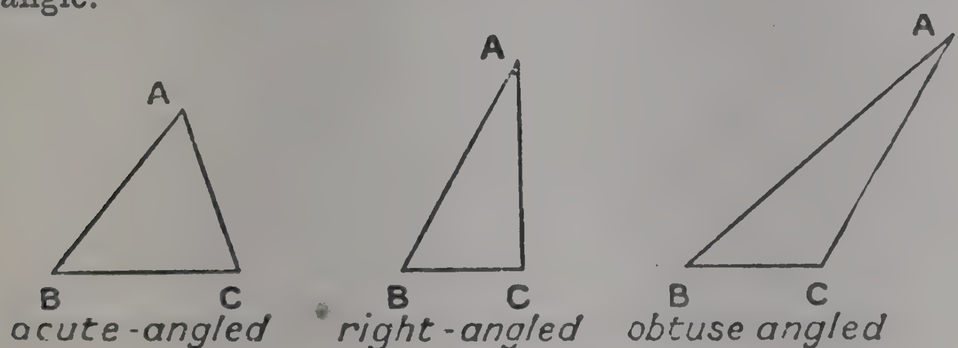


FIG. 34.

A triangle, *all* of whose angles are acute, is called **acute-angled**.

A triangle, *one* of whose angles is a right angle, is called **right-angled**.

A triangle, *one* of whose angles is obtuse, is called **obtuse-angled**.

The lines AB, BC, CA are called the **sides** of the triangle ABC.

The points A, B, C are called the **vertices** of the triangle ABC.

If any corner, say B, is called the **vertex** of the triangle ABC, then the opposite side AC is called the **base**.

If a triangle is right-angled, the side opposite the right-angle is called the **hypotenuse.**

A plane figure bounded by four straight lines is called a **quadrilateral.**

If ABCD is a quadrilateral, the lines AC, BD are called its **diagonals.**

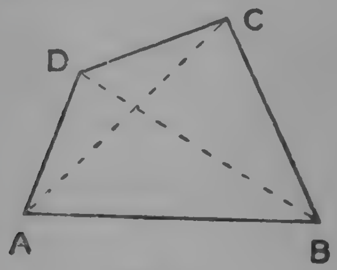


FIG. 35.

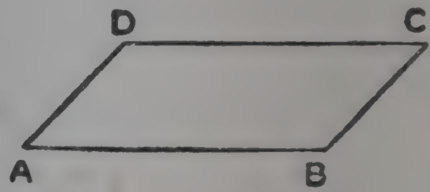


FIG. 36.

A quadrilateral whose opposite sides are parallel is called a **parallelogram.**

A plane figure bounded by any number of straight lines is called a **polygon.** If each interior angle is less than two right angles, the polygon is called a **convex polygon.**

If one or more of the interior angles of a polygon are reflex, the polygon is called **a re-entrant polygon.**

FIG. 37.

The following special names are also used :

5 sides	6 sides	7 sides	8 sides	10 sides
pentagon	hexagon	heptagon	octagon	decagon

The sum of the lengths of the sides of a polygon is called its **perimeter.**

A polygon is called **equiangular** if all its angles are equal.

A polygon is called **equilateral** if all its sides are of equal length.

A polygon is called **regular** if all its angles are equal and all its sides are of equal length.

EXERCISE XII. (Oral).

1. Draw any triangle ABC. Place your pencil along BC ; hold it at C, and turn it through the angle BCA so that it now lies along CA. Next hold it at A, and turn it through the angle CAB so that it lies along AB. Lastly hold it at B and turn it through the angle ABC so that it lies along BC.
   (i) Does it end in the same direction as it started ?
   (ii) What is the total size of the angle turned through ?
   (iii) What are the separate angles turned through ?
   (iv) What conclusion do you draw from the process ?

2. Take a large triangle ABC on the floor. Start at the mid-point of BC and walk to C, now turn and walk to A, then turn and walk to B, and finally turn and walk back to where you started.
   (i) Draw on your paper a plan of how you have walked, and mark on it the various angles through which you have turned.
   (ii) What is the total amount you have turned when you have returned to the starting point ?
   (iii) What conclusion do you draw from the process ?
   (iv) How can you find the sum of the angles of the triangle ABC from this conclusion ?

3. Draw a figure like Fig. 38. Place your pencil along BCD, hold it at B and turn it anti-clockwise so that it lies along BA ; now hold it at A and turn it anti-clockwise so that it lies along AC.
   (i) What are the two angles through which you have turned it?
   (ii) What angle in the figure represents the difference of direction of its initial and final direction ?
   (iii) What conclusion can you draw from the process ?

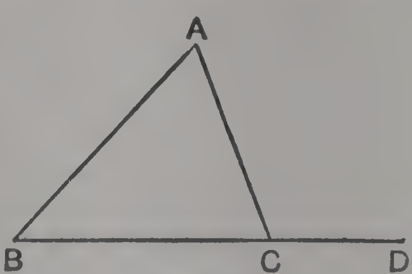


FIG. 38.

4. Cut out a paper triangle. Tear off the three corners and fit them together so that the corners are at the same point. What do you notice and what conclusion can you draw ?

5. Cut out a paper quadrilateral, tear off the four corners and fit them together at one point. What do you notice and what follows ?

6. Use your protractor to measure the angles of the triangle ABC in Fig. 8, page 7. Find their sum. Do this also for each of the triangles DEF and KLM in that figure.

What fact does this illustrate ?

7. Imagine the process described in question 2 was carried out with a very big triangle. Suppose one corner is the North Pole and the other two corners are on the equator. How would this affect the argument and the conclusions derived from it ?

8. A field is in the shape of a convex polygon, bounded by (say) seven straight unequal fences. A man starts at the middle of one fence and walks all round the outside of the field till he returns to his starting-point. Draw a figure to represent the field and mark on it the angles through which the man turns at the corners. What is the total angle he has turned through during his walk ? What conclusion can you draw from this process ?

What is the sum of the interior angles of the polygon ?

9. Draw on your paper a convex pentagon ABCDE. Place your pencil flat on the paper along AB, with the sharpened point at B. Hold it at B and turn it through the angle ABC ; transfer your hold to C and continue the process, turning it about each corner in turn, until the pencil again lies along AB. What is the total angle turned through by the pencil and what conclusion can you draw ?

10. If you perform the process described in question 9 for a figure of 10 sides, how many complete turns will the pencil make before it is again lying along the line on which it started ?

11. Draw any triangle ABC and produced BC to any point D. Through C draw a line CF parallel to BA (see Fig. 39).

(i) State what alternate angles are equal.
(ii) State what corresponding angles are equal.
(iii) What can you say about the angle ACD ?
(iv) What can you say about the angles of the triangle ABC ?

12. Draw any seven-sided polygon ABCDEFG, and mark any point O inside the polygon ; join O to each corner of the polygon.

(i) How many triangles are there in the figure ?
(ii) What is the sum of the angles of all these triangles ?
(iii) What is the sum of all the angles at O ?
(iv) What is the sum of the interior angles of the polygon ?
(v) Produce each side of the polygon in order, going round the figure anti-clockwise. What is the sum of the exterior angles so formed ?

The facts which the last Exercise illustrate are stated and proved formally in the next two theorems.

THEOREM 6.

(1) *If a side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.*

(2) *The sum of the three angles of any triangle is two right angles.*

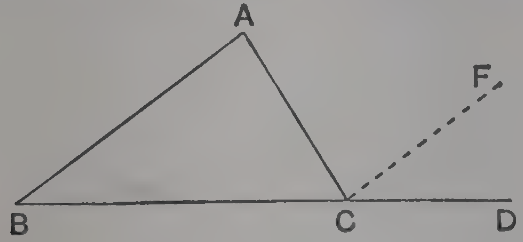


FIG. 39.

ABC is a triangle ; BC is produced to D,

To prove (1) $\angle ACD = \angle CAB + \angle ABC$.

(2) $\angle CAB + \angle ABC + \angle ACB = 180°$.

(1) Let CF be drawn parallel to AB.

$\angle FCD = \angle ABC$, corresp. angles. $\angle ACF = \angle CAB$, alt. angles.

adding, $\angle FCD + \angle ACF = \angle ABC + \angle CAB$.

$\therefore\ \angle ACD = \angle ABC + \angle CAB$.

(2) Add to each the angle ACB.

$\therefore\ \angle ACD + \angle ACB = \angle ABC + \angle CAB + \angle ACB$.

But $\angle ACD + \angle ACB = 180°$, adj. angles, BCD a st. line.

$\therefore\ \angle ABC + \angle CAB + \angle ACB = 180°$. Q.E.D.

Corollary 1. *If two triangles have two angles of the one equal to two angles of the other, each to each, then the third angles are also equal.*

This follows from the fact that the sum of all three angles of the first triangle is equal to the sum of all three angles of the second, since each sum is two right angles.

Corollary 2. *In a right-angled triangle* (i) *the right angle is the greatest angle* ; (ii) *the sum of the remaining angles is equal to a right angle.*

Since the sum of the three angles of a triangle is two right angles, if one angle is a right angle, the sum of the other two angles is a right angle, and therefore each of these two angles separately is less than a right angle.

THEOREM 7.

(1) **All the interior angles of a convex polygon, together with four right angles, are equal to twice as many right angles as the polygon has sides.**

(2) **If all the sides of a convex polygon are produced in order, the sum of the exterior angles is four right angles.**

Let n be the number of sides of the polygon.

(1) *To prove* that

the sum of the angles of the polygon $+ 4$ rt. $\angle$s $= 2n$ rt. $\angle$s.

Take any point O inside the polygon and join it to each vertex.

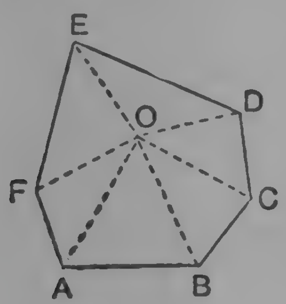


FIG. 40 (i).

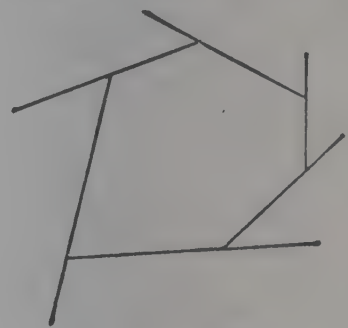


FIG. 40 (ii).

The polygon is now divided into n triangles.

But the sum of the angles of each triangle is 2 rt. $\angle$s.

$\therefore$ the sum of the angles of the n triangles is $2n$ rt. $\angle$s.

But these angles make up all the angles of the polygon together with all the angles at O.

Now the sum of all the angles at O is 4 rt. $\angle$s.

$\therefore$ all the angles of the polygon $+ 4$ rt. $\angle$s $= 2n$ rt. $\angle$s.

(2) At each vertex, the interior $\angle$ + the exterior $\angle = 2$ rt. $\angle$s.

$\therefore$ the sum of all the interior angles + the sum of all the exterior angles $= 2n$ rt. $\angle$s.

But the sum of all the interior angles $+ 4$ rt. $\angle$s $= 2n$ rt. $\angle$s.

$\therefore$ the sum of all the exterior $\angle$s $= 4$ rt. $\angle$s. Q.E.D.

Theorem 7 (1) may also be stated as follows:

The sum of the interior angles of any convex polygon of n sides is $2n - 4$ right angles.

EXERCISE XIII.

1. In a right-angled triangle, one angle is 37°, what is the third angle ?

2. Two angles of a triangle are each 53°, what is the third angle ?

3. If $\angle BAC = 43°$ and $\angle ABC = 109°$, what is $\angle ACB$?

4. The side BC of the triangle ABC is produced to D ; $\angle ABD = 19°$, $\angle ACD = 37°$, what is $\angle BAC$?

5. In the quad. ABCD, $\angle ABC = 112°$, $\angle BCD = 75°$, $\angle DAB = 51°$, what is $\angle CDA$?

6. ABCD is a straight line and P a point outside it ; $\angle PBA = 110°$, $\angle PCD = 163°$, find $\angle BPC$.

7. Three of the angles of a quad. are equal ; the fourth angle is 120° ; find the others.

8. Can a triangle be drawn having its angles equal to (i) 43°, 64°, 73° ; (ii) 45°, 65°, 80° ?

9. What is the remaining angle of a triangle, if two of its angles are (i) 120°, 40° ; (ii) 50°, $x°$; (iii) $2x°$, $3x°$; (iv) $x+10$, $20-x$ degrees ?

10. The angles of a triangle are $x°$, $2x°$, $2x°$; find x.

11. If in the triangle ABC, $\angle BAC = \angle BCA + \angle ABC$, find $\angle BAC$.

12. If A, B, C are the angles of a triangle and if $A - B = 15°$, $B - C = 30°$, find A.

13. The angles of a five-sided figure are x, $2x$, $x+30$, $x-10$, $x+40$ degrees, find x.

14. The angles of a pentagon are in the ratio $1 : 2 : 3 : 4 : 5$; find them.

15. In $\triangle ABC$, $\angle ABC = 38°$, $\angle ACB = 54°$; AD is perpendicular to BC ; AE bisects $\angle BAC$; find $\angle EAD$.

16. In $\triangle ABC$, $\angle BAC = 74°$, $\angle ABC = 28°$; BC is produced to X ; the lines bisecting $\angle ABC$ and $\angle ACX$ meet at K ; find $\angle BKC$.

17. In $\triangle ABC$, $\angle ABC = 32°$, $\angle BAC = 40°$; find the angle at which the bisector of the greatest angle of the triangle cuts the opposite side.

18. In $\triangle ABC$, $\angle ABC = 110°$, $\angle ACB = 50°$; AD is the perpendicular from A to CB produced ; prove that $\angle DAB = \angle BAC$.

19. The base BC of △ABC is produced to D; if ∠ABC=∠ACB and if ∠ACD=$x°$; calculate ∠BAC.

20. In the quad. ABCD, ∠ABC=140°, ∠ADC=20°; the lines bisecting the angles BAD, BCD meet at O; calculate ∠AOC.

21. In △ABC, the bisector of ∠BAC cuts BC at D, if ∠ABC=$x°$ and ∠ACB=$y°$; calculate ∠ADC.

22. If the angles of a quad. taken in order are in the ratio 1 : 3 : 5 : 7; prove that two of its sides are parallel.

23. Each angle of a polygon is 140°; how many sides has it ?

24. Find the sum of the interior angles of a 12-sided convex polygon.

25. Find the interior angle of a regular 20-sided figure.

26. Prove that the sum of the interior angles of an 8-sided convex polygon is twice the sum of those of a pentagon.

27. Each angle of a regular polygon of x sides is $\frac{3}{4}$ of each angle of a regular polygon of y sides; express y in terms of x, and find any values of x, y which will fit.

28. The sum of the interior angles of an n-sided convex polygon is double the sum of the exterior angles. Find n.

29. In Fig. 41, prove that $x=a+b-y$.

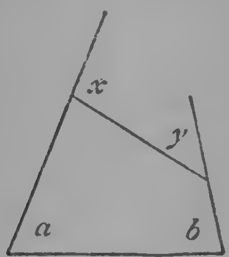


FIG. 41.

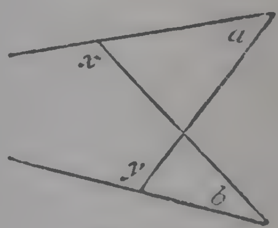


FIG. 42.

30. In Fig. 42, prove that $x-y=a-b$.

31. In Fig. 43, express x in terms of a, b, c.

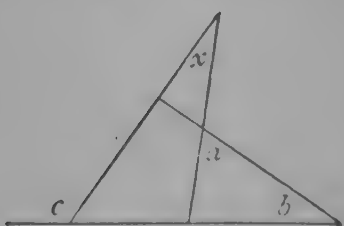


FIG. 43.

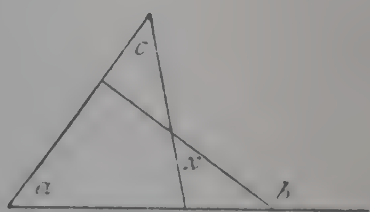


FIG. 44.

32. In Fig. 44, express x in terms of a, b, c.

33. In Fig. 45, express x in terms of a, b, c.

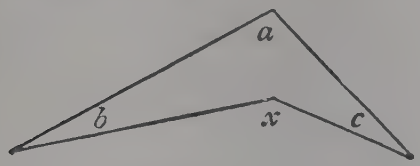


FIG. 45.

34. If, in Fig. 46, $x+y=3z$, prove that the triangle is right-angled.

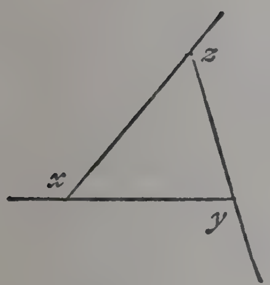


FIG. 46.

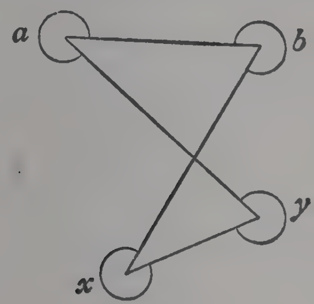


FIG. 47.

35. Prove that the reflex angles in Fig. 47 are connected by the relation $a+b=x+y$.

36. D is a point on the base BC of the triangle ABC such that $\angle DAC=\angle ABC$; prove that $\angle ADC=\angle BAC$.

37. The diagonals of the parallelogram ABCD meet at O; prove that $\angle AOB=\angle ADB+\angle ACB$.

38. If, in the quadrilateral ABCD, AC bisects the angle DAB and the angle DCB; prove that $\angle ADC=\angle ABC$.

39. ABC is a triangle, right-angled at A; AD is drawn perpendicular to BC; prove that $\angle DAC=\angle ABC$.

40. In the $\triangle ABC$, BE and CF are perpendiculars from B, C to AC, AB; BE cuts CF at H; prove that $\angle CHE=\angle BAC$.

41. If, in the quadrilateral ABCD, $\angle ABC=\angle ADC$ and $\angle BCD=\angle BAD$; prove that ABCD is a parallelogram.

42. If in the $\triangle ABC$ the bisectors of the angles ABC, ACB meet at I; prove that $\angle BIC=90°+\frac{1}{2}\angle BAC$.

43. The side BC of $\triangle ABC$ is produced to D; the lines bisecting $\angle ABC$, $\angle ACD$ meet at Q; prove that $\angle BQC=\frac{1}{2}\angle BAC$.

44. The base BC of $\triangle ABC$ is produced to D; the bisector of $\angle BAC$ cuts BC at K; prove $\angle ABD+\angle ACD=2\angle AKD$.

45. In the quadrilateral ABCD, the lines bisecting $\angle ABC$, $\angle BCD$ meet at P; prove that $\angle BAD+\angle CDA=2\angle BPC$.

The Use of Compasses.

A **circle** is a plane figure bounded by a curved line, all points of which are equidistant from a fixed point, called the **centre** of the circle. The curved line is called the **circumference** of the circle. The distance between the centre and any point on the circumference is called the **radius**.

The straight line joining any two points on the circumference of a circle is called a **chord**.

A chord which passes through the centre of the circle is called a **diameter**.

Any portion of the circumference is called an **arc**.

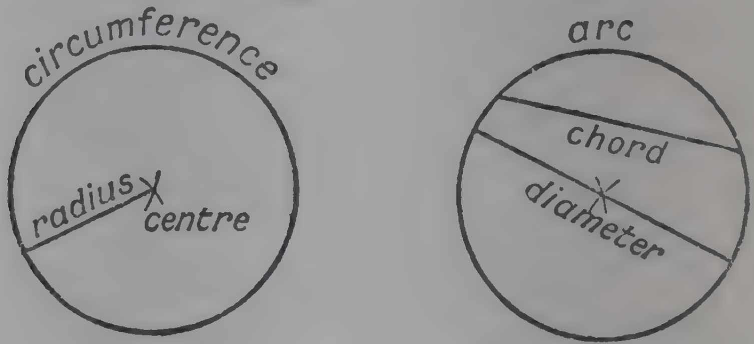


FIG. 48.

If two circles have the same centre, they are called **concentric**.

By using a pair of compasses, it is possible to draw a circle having any given point as centre and any given length as radius.

Compasses may also be used to cut off from a given straight line any given length.

To draw a circle on a given line AB as diameter, find the middle point O of AB, and then describe a circle with O as centre and OA as radius.

EXERCISE XIV.

1. Draw two concentric circles of radii 4 cm., 6 cm. Draw any diameter of the larger circle. What are the lengths of the three parts into which it is cut by the smaller circle ?

2. Draw two circles of radii 3 cm., 4 cm., so that their centres are 5 cm. apart. Draw their common chord, *i.e.* the line joining the points at which they cut. Measure its length.

3. Take two points A, B, 4 cm. apart; construct two points P, Q such that PA=PB=5 cm.=QA=QB.

4. Take a point P; construct a circle of radius 4 cm. passing through P. Construct a chord PQ of length 6 cm.

5. Draw a large triangle ABC with unequal sides. Draw the circles whose diameters are AB and AC. Do they cut each other on BC?

6. Take two points A, B 5 cm. apart. Construct a point C such that CA=6 cm., CB=7 cm. Is there more than one possible position for C?

7. Draw a circle of radius 4 cm. and place in it six chords, end to end, each 4 cm. long. What figure is obtained?

8. Draw two circles of radii 4 cm., 5 cm., so that the part of the line joining their centres which lies inside both circles is of length 2 cm.

9. Draw a line AB 3 cm. long; construct a circle of radius 4 cm. to pass through A and B.

10. Take two points A, B 6 cm. apart. Construct 10 positions of a point P on either side of AB, such that PA+PB=10 cm. (*e.g.* PA=3, PB=7 or PA=4, PB=6, etc.). All these positions of P lie on a smooth curve called an **ellipse**. Draw freehand a curve through these positions. Would you expect the curve to pass through A or B?

11. Construct a triangle so that its sides are 2 in., 3 in., 4 in. long. Are these measurements sufficient to fix the size of the triangle? Measure the largest angle.

12. Construct a triangle so that two of its sides are 2·5 in. and 3·5 in. long, and the angle between these sides is 107°. Are these measurements sufficient to fix the size of the triangle? Measure the third side.

13. Construct a quadrilateral ABCD such that AB=5 cm., BC=4 cm., CD=7 cm., DA=6 cm., ∠ABC=110°. Measure ∠ADC.

14. Construct a quadrilateral ABCD such that AB=5 cm., BC=4 cm., CD=8 cm., AD=9 cm., BD=10 cm. Measure ∠ABC.

Data necessary to fix the Size and Shape of a Triangle.

Suppose we wish to make an exact copy of the triangle ABC, Fig. 8, p. 7. What is the least number of measurements that

must be taken to fix it? There are 3 sides and 3 angles. Is it necessary to measure all these six things? If not, what selection of them is sufficient?

EXERCISE XV. (Oral).

(The following questions refer to Fig. 8 on p. 7.)

1. Measure AB, AC and the angle BAC. Are these sufficient for making a copy of the triangle ABC? If so, construct the copy, measure the line BC in your copy. (The average of the results obtained by the class should be taken and compared with the length of the original.)

2. Make a copy of the triangle DEF by measuring ED, EF and the angle DEF, and compare results as in question 1.

Hence we conclude that the size and shape of a triangle are fixed if the lengths of two sides and the size of the angle included by those two sides are known.

In other words, if two triangles are drawn so that they agree as regards the measurements of two separate sides and the included angle, then they must agree as regards all other corresponding measurements. Such triangles are said to be equal in all respects or congruent. (*The sign for congruent is* $\equiv$.)

3. Measure BC, $\angle$ABC, $\angle$ACB. Can you construct a copy of the triangle ABC with these measurements? If so, do it and measure the line AB in your copy. Compare results as in question 1.

4. Measure KL, $\angle$LKM, $\angle$KML. Can you say what $\angle$KLM is without measuring it? Construct the triangle KLM without any more measurements. Measure LM, and compare results as in question 1.

Hence we conclude that the size and shape of a triangle are fixed if the sizes of two angles and the length of a side of known situation with respect to these angles are given.

In other words, if two triangles are drawn so that they agree as regards the measurements of two separate angles and one side correspondingly situated *in the two triangles, then they must agree as regards all other* corresponding *measurements, i.e. they must be congruent.*

5. Draw two triangles which are obviously not congruent but yet have two angles of the first respectively equal to two angles of the second, and also one side of the first equal to one side of the second.

6. The two triangles in Fig. 49 are congruent.

(i) Which side corresponds to AX, and why ?

(ii) Which angle corresponds to ∠XAQ, and why ?

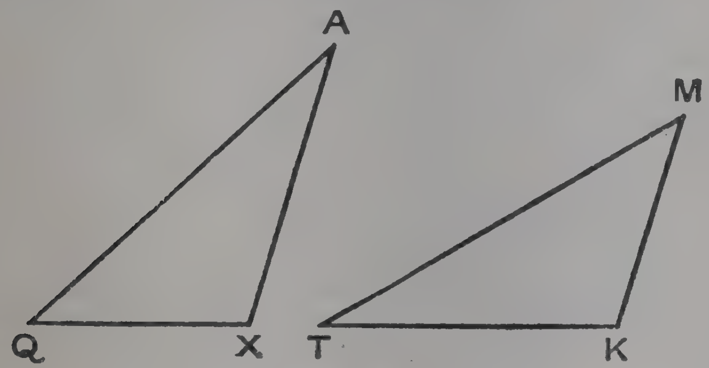


FIG. 49.

7. Construct a triangle ABC so that AB=AC. It can be proved (or you can see from symmetry) that the angles at B and C are equal. Now take any point X on BC nearer to B than to C.

What sides and angles of the triangle ABX are equal to sides and angles of the triangle ACX ? Are these two triangles congruent ?

8. Measure AB, BC, CA. With these measurements copy the triangle ABC and measure ∠ACB. Compare results as in question 1.

9. Repeat question 8 for the triangle KLM.

We conclude that if two triangles are drawn so that they agree as regards the measurements of all three sides, then they must agree as regards all other corresponding *measurements, i.e. they must be congruent.*

10. Measure the three angles of the triangle KLM. Can you make a copy of the triangle without any further measurements ? When you had measured two of the angles was it necessary to measure the third angle ? Can you draw two triangles which are obviously not congruent but yet agree, as regards all three angles, with these measurements ? What do you notice about the shape of the two triangles ?

We conclude that when the angles of a triangle are given the shape *of the triangle is fixed, but its size is not determined until the length of one side is also given.*

11. Construct a triangle the same shape as DEF but of a different size.

12. Can you construct two quadrilaterals which are obviously not the same shape but such that the four angles of one are respectively equal to the four corresponding angles of the other ?

Is the shape of a quadrilateral fixed when all its angles are given in size ?

13. A triangle ABC is to be copied : the following measurements are given, AB=7 cm., AC=5·5 cm., ∠ABC=35°. Are these measurements sufficient to fix the size of the triangle ? If so, construct it ; if not, construct triangles of different sizes which fit these measurements. Measure BC and compare results.

14. A triangle ABC is to be copied : the following measurements are given, AB=7 cm., AC=8 cm., ∠ABC=35°. Are these measurements sufficient to fix the size of the triangle ? If so, construct it ; if not, construct triangles of different sizes which fit these measurements. Measure BC and compare results.

15. Construct a triangle ABC so that AB=AC ; produce CB to any point X and join AX. What sides and angles of the triangle AXB are equal to the corresponding sides and angles of the triangle AXC. Are the triangles congruent ?

16. Draw two triangles ABC, XYZ which are obviously not congruent, but for which AB=XY, AC=XZ, ∠ABC=∠XYZ. Is one of the triangles obtuse-angled, and must this be so ? Is AB greater than AC, and must this be so ?

17. A triangle ABC is right-angled at B, AB=5 cm., AC=7 cm. Are these measurements sufficient to fix the size of the triangle ? Construct the triangle ABC, measure BC and compare results as in question 1.

We conclude that the size and shape of a triangle are not always fixed **uniquely** *when the measurements of two sides and a* **not-included angle** *are given.*

If, however, the triangle is right-angled, the measurement of two sides is sufficient to fix its size and shape. In other words, if two **right-angled** *triangles are drawn so that they agree as regards the lengths of their hypotenuses and one other pair of sides, then they must agree as regards all other corresponding measurements, i.e. they must be congruent.*

Example 14 illustrates the more general statement that the measurement of two sides and a not-included angle fixes the

size of the triangle uniquely, provided that the given not-included angle is opposite to the greater of the two given sides, but not otherwise.

For further examples on the construction of triangles, quadrilaterals, etc., with numerical data, see p. 80.

The conclusions of this Exercise may be summarised as follows :

The size and shape of a triangle are fixed by any of the following sets of measurements :

(i) Two sides and the included angle.

(ii) One side and two angles, given the situation of the side with respect to the angles.

(iii) Three sides.

Further :

(iv) If the triangle is right-angled, the hypotenuse and one other side.

But an ambiguous case arises if two sides and a not-included angle are measured. Two triangles can be drawn to fit these measurements if the given angle is opposite the shorter of the two given sides.

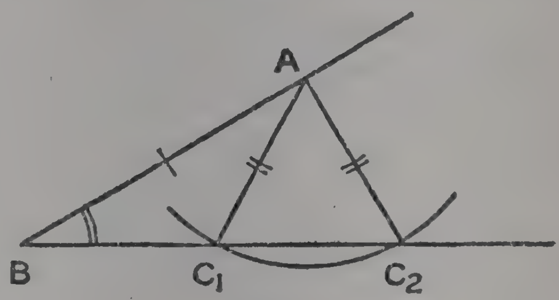


FIG. 50.

Fig. 50 shows the construction for drawing a triangle if two sides and a not-included angle are given. Fig. 51 shows the

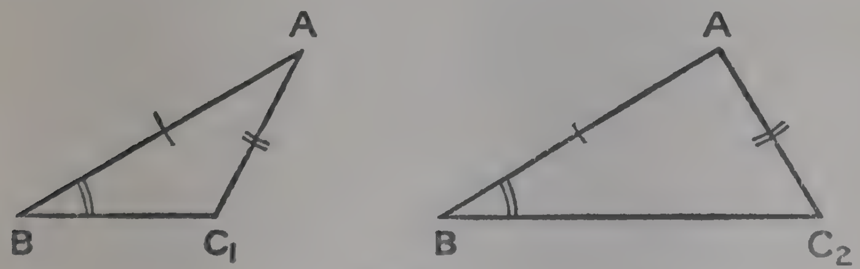


FIG. 51.

two different kinds of triangles that can be drawn to fit the given measurements.

The tests for congruence given above are expressed in the following theorems.

THEOREM 8.

In the triangles ABC, PQR,

If $AB = PQ$, $AC = PR$, $\angle BAC = \angle QPR$

Then $\triangle ABC \equiv \triangle PQR$.

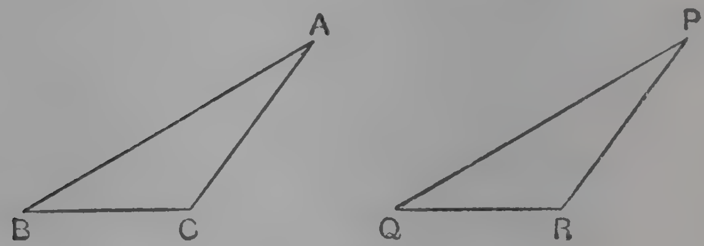


FIG. 52.

THEOREM 9.

In the triangles ABC, PQR,

(i) If $BC = QR$, $\angle ABC = \angle PQR$, $\angle ACB = \angle PRQ$

Then $\triangle ABC \equiv \triangle PQR$.

(ii) If $BC = QR$, $\angle ABC = \angle PQR$, $\angle BAC = \angle QPR$,

Then $\triangle ABC \equiv \triangle PQR$.

THEOREM 10.

In the triangles ABC, PQR.

If $AB = PQ$, $BC = QR$, $CA = RP$,

Then $\triangle ABC \equiv \triangle PQR$.

THEOREM 12.

In the triangles ABC, PQR,

If $\angle ABC = 90° = \angle PQR$, $AC = PR$, $AB = PQ$,

Then $\triangle ABC \equiv \triangle PQR$.

(A formal proof of Theorem 12 will be found on page 55.)

EASY EXAMPLES ON CONGRUENCE.

State a reason for each step in the argument.

EXERCISE XVI.

1. If the straight line XOY bisects at right angles the straight line AOB, prove that XA=XB.

2. Two unequal straight lines AOB, COD bisect each other; prove that AC=BD.

3. ABC is a triangle such that AB=AC; the line bisecting the angle BAC cuts BC at E; prove that (i) $\angle$ABC=$\angle$ACB, (ii) E is the mid-point of BC, (iii) AE is perpendicular to BC.

4. ABCD is a quadrilateral such that AB is equal and parallel to DC; prove that AD is equal and parallel to BC.

5. A line AP is drawn bisecting the angle BAC; PX, PY are the perpendiculars from P to AB, AC; prove that PX=PY.

6. D is the mid-point of the base BC of the triangle ABC, BX, CY are the perpendiculars from B, C to the line AD, produced if necessary; prove that BX=CY.

7. A straight line cuts two parallel lines at A, B; C is the mid-point of AB; any line is drawn through C cutting the parallel lines at P, Q; prove that PC=CQ.

8. The diagonals of a quadrilateral bisect each other at right angles, prove that all its sides are equal.

9. ABCD is a quadrilateral such that AB is parallel to DC and AD is parallel to BC; prove that AB=DC and AD=BC. (Construction: join AC.) Prove also that AC bisects BD.

10. X is the mid-point of a chord AB of a circle, centre O; prove that $\angle$OXA=90°.

11. ABCD is a quadrilateral such that AB=CD and AD=BC; prove that AD is parallel to BC. (Join AC.)

12. AB and CD are two equal chords of a circle, centre O; prove that $\angle$AOB=$\angle$COD.

13. ABC and XYZ are two triangles. By marking the data on a figure, find out which of the following sets of data make the two triangles congruent. Give reasons.

(i) AB=YZ, AC=XZ, $\angle$A=$\angle$Z.
(ii) AB=XY, $\angle$A=$\angle$X, $\angle$B=$\angle$Z.
(iii) AC=XZ, AB=XY, $\angle$C=$\angle$Z.

(iv) AB = YZ, $\angle A = \angle Z$, $\angle C = \angle X$.
(v) AB = AC, XY = XZ, $\angle A = \angle X$.
(vi) BC = ZX, AC = YX, BC = YZ.
(vii) $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.
(viii) AC = YZ, BC = XY, $\angle C = \angle Y$.
(ix) BC = XZ, $\angle A = \angle Y$, $\angle B = \angle X$.
(x) AB = YZ, BC = XY, $\angle A = \angle Z$.

14. What additional fact about the remaining sides or angles must be given to prove the triangles ABC, XYZ congruent in the following cases? Give as many alternative answers as possible.

(i) AB = YZ, $\angle A = \angle Z$.
(ii) BC = XZ, AB = XY.
(iii) $\angle B = \angle X$, $\angle C = \angle Y$.
(iv) BC = XZ, $\angle A = \angle Y$.

15. Draw a triangle ABC so that AB is greater than AC. Take the middle point D of BC and draw a line DX through D perpendicular to BC. Draw the line which bisects the angle BAC and let it cut DX at X. From X draw lines XP, XQ perpendicular to AB, AC and cutting AB, AC produced if necessary at P, Q. Join XB, XC.

(i) Find in the figure three pairs of congruent triangles and prove the congruence.

(ii) Prove that AP = AQ and BP = CQ. How is this reconciled with the fact that AB is greater than AC?

Triangles of the Same Shape but Different Size.

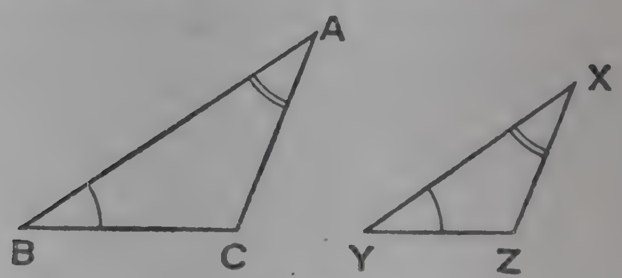


FIG. 53.

If in the two triangles ABC, XYZ, we are given

$$\angle B = \angle Y \quad \text{and} \quad \angle C = \angle Z,$$

we know that $\angle A$ must also equal $\angle X$, because the three angles of every triangle add up to two right angles.

But the triangles are not congruent unless there is an equal pair of corresponding sides.

If the triangles are equiangular but not congruent, they are of the same shape but different size. We can regard in Fig. 53, the triangle XYZ as a model of the triangle ABC on a reduced scale.

The ratio $\frac{XY}{AB}$ represents the scale on which the reduction has been made, and must be the same as the ratios $\frac{YZ}{BC}$ and $\frac{XZ}{AC}$ in which other lines in the figure have been reduced. The triangle ABC in this case is said to be similar to the triangle XYZ.

This is the principle used in constructing plans of houses or maps of small areas of ground. The shape of the plan is the same as that of the house it represents, and all lengths in the plan are reduced in a fixed ratio from the actual corresponding length in the building. This fixed ratio is called the scale of the plan or map.

When drawing a plan to scale, start by making a rough sketch, putting in the given measurements. Then choose a convenient scale and *state what your scale is.*

DRAWING TO SCALE.

Definitions. (i) In Fig. 54, if OA is horizontal, ∠AOB is called **the angle of elevation** of B as viewed from O.

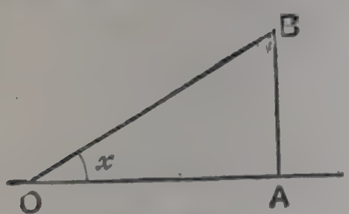


Fig. 54.

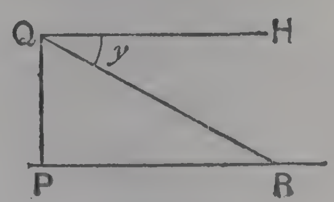


Fig. 55.

(ii) In Fig. 55, if QH is horizontal, ∠HQR is called the angle of depression of R as viewed from Q.

EXERCISE XVII.

1. A courtyard is 80 feet long and 50 feet wide; what is the distance between two opposite corners?

2. A gun whose range is 5000 yards is in position at a point 3500 yards from a straight railway line; what length of the line can it command?

3. A ladder, 15 feet long, is resting against a vertical wall; the foot of the ladder is 6 feet from the wall; how high up the wall does it reach?

4. The ends of a cord, 10 feet long, are fastened to two nails each of which is 15 feet above the ground; the nails are 5 feet apart; a weight is attached to the mid-point of the cord; how high is it above the ground?

5. A straight passage runs from A to B, then turns through an angle of 70° and runs on to C; if AB is 80 yards and BC is 100 yards, what distance is saved by having a passage direct from A to C?

6. A man rows due North at 4 miles an hour, and the current takes him North-east at 5 miles an hour; how far is he from his starting-point after 20 minutes?

7. A man starts from A and walks 2 miles due South to B, then 3 miles South-west to C, then 1 mile West to D; what is the direction and distance of D from A?

8. Southampton is 12 miles S.S.W. of Winchester; Romsey is 10 miles W. 32° S. of Winchester. Find the distance and bearing of Romsey from Southampton.

9. An aeroplane points due North and flies at 60 miles an hour; the wind carries it S.W. at 15 miles an hour. What is its position ten minutes after leaving the aerodrome?

10. Andover is 12 miles from Winchester and 15 miles from Salisbury; Salisbury is 20 miles W. of Winchester. (Andover is north of the Salisbury-Winchester line.) Find the bearing of Andover from Salisbury.

11. Exeter is 42 miles from Dorchester and 64 miles from Bristol; Bristol is 55 miles due North of Dorchester; Barnstable is 33 miles N.E. of Exeter. What is the distance and bearing of Barnstable from Dorchester?

12. From two points 500 yards apart on a straight road running due North, the bearings of a house are found to be N. 40° E. and E. 20° S.; find the shortest distance of the house from the road.

13. One end of a string, 5 feet long, is fastened to a nail, and a weight is attached to the other end; the weight swings backwards and forwards through 15° each side of the vertical. What is the distance between its two extreme positions?

14. At a distance of 40 yards from a tower, the angle of elevation of the top of the tower is 35° ; find the height of the tower in feet.

15. A kite is flown at the end of a string 120 yards long which makes an angle of 65° with the ground ; find in feet the height of the kite.

16. What is the elevation of the sun when a pole 12 feet high casts a shadow 20 feet long ?

17. From the top of a cliff 150 feet high, the angle of depression of a boat out at sea is 20° ; what is the distance of the boat from the cliff in yards ?

18. From the top of a tower 250 feet high, the angles of depression of two houses in a line with and at the same level as the foot of the tower are 64° and 48°. Find their distance apart in yards.

The Trigonometrical Ratios.

Opinion differs as to when it is best to introduce the ideas of this section. If taken at an early stage it affords practical applications for work in multiplication and division of decimals. But it can be postponed without interfering with the remainder of the geometry course.

The Tangent of an Angle.

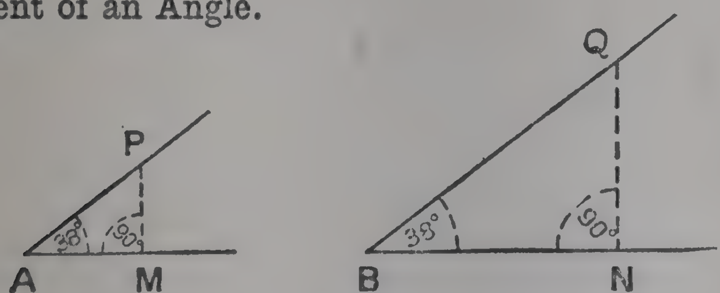


FIG. 56.

Draw two angles A, B each 38°, and from points P, Q on one arm of each angle draw the perpendiculars PM, QN to the other arm.

Then the triangles PAM, QBN are the same shape, and we may regard the triangle PAM as a scale drawing of the triangle QBN. $\therefore$ the ratio $\frac{PM}{AM}$ equals the ratio $\frac{QN}{BN}$.

$\therefore$ the value of the ratio $\frac{PM}{AM}$ does not depend on the distance of P from the point A, but only on the size of the angle PAM.

The ratio $\frac{PM}{AM}$ is called the **tangent of the angle** PAM, and is written tan PAM or tan 38°, since $\angle PAM = 38°$.

In words, if a perpendicular is let fall from any point in one arm of the angle to the other arm, the tangent of the angle

$$= \frac{\text{opposite side (PM)}}{\text{adjacent side (AM)}}.$$

We may find the approximate value of the tangent of any angle by measurement and computation.

Here $\tan 38 = \frac{QN}{BN} = \frac{1{\cdot}7}{2{\cdot}2}$ approx. $= 0{\cdot}78$ approx.

The tangents of angles are calculated, once for all, and entered in a table from which they can be obtained when needed.

It should be noted that, in Fig. 56 $\angle NQB = 90° - 38° = 52°$;

$$\therefore \frac{BN}{QN} = \tan 52°.$$

EXERCISE XVIII.

(For Tables, see p. 52.)

1. Find by drawing and measurement approximate values of tan 25°, tan 45°, tan 60°, tan 72°.

2. Find by drawing and measurement the angles whose tangents are 0·25, 0·70, 1·28, 2.

3. In Fig. 56, if BN = 7 cm., $\angle B = 35°$, find QN.

4. In Fig. 56, if QN = 8 cm., $\angle BQN = 62°$, find BN.

5. In Fig. 56, if QN = 8 cm., $\angle QBN = 56°$, find BN.

6. In Fig. 56, if QN = 7 cm., BN = 8 cm., find $\angle B$.

7. In Fig. 56, if BN = 3·4 cm., QN = 10 cm., find $\angle B$.

8. A man standing 100 ft. from the foot of a tower observes the angle of elevation of the top of the tower is 52°. What is the height of the tower?

9. What is the angle of elevation of the top of a tower 126 feet high from a point on the ground 90 feet from the foot of the tower ?

10. A man on the top of a cliff 180 feet high sees that the angle of depression of a boat is 27°. How far is the boat from the foot of the cliff ?

11. A man starts from O and walks 320 yards East and then 140 yards North. What is now his bearing from O ?

12. The shadow of a telegraph pole is 18 feet long when the altitude of the sun is 58°. What is the height of the pole ?

13. What is the elevation of the sun if a stick 5 feet long casts a shadow 3 feet 9 inches long ?

14. The pole of a bell tent is 8·2 feet long and the diameter of the base of the tent is 14 feet. What angle does the slant side of the tent make with the ground ?

15. Two men stand on opposite sides of a tower 85 feet high and measure the angle of elevation of the top of the tower as 25° and 33° respectively. How far apart are the two men ?

16. The steps of a staircase are 11 inches wide and 6 inches high. What is the slope of the staircase, in degrees ?

17. A man stands at a distance of 85 feet from the foot of a building and observes that the angles of elevation of the top and bottom of a flagstaff on its roof are 56° and 54° respectively. What is the length of the flagstaff ?

18. A rectangular sheet of paper is 9 inches long, 7 inches wide. What angle does a diagonal of the sheet make with the longer side ? At what angle do the two diagonals cut each other ?

19. A man standing on a cliff 160 feet high observes two boats in a vertical plane with him. The angles of depression of the boats are 32°, 49°. What is the distance between the boats ?

20. The shadow of a telegraph pole is 13 feet long when the elevation of the sun is 59°. What is the length of the shadow when the sun's elevation is 35° ?

The Cosine of an Angle.

In Fig. 56, since the triangle PAM is a scale drawing of the triangle QBN, the ratio $\frac{AM}{AP}$ equals the ratio $\frac{BN}{BQ}$, and so the value of the ratio $\frac{AM}{AP}$ does not depend on the distance of P from the point A, but only on the size of the angle PAM.

The ratio $\frac{AM}{AP}$ is called the cosine of the angle PAM, and is written cos PAM or cos 38°, if $\angle PAM = 38°$.

In words, if a perpendicular is let fall from any point in one arm of an angle to the other arm, the cosine of the angle

$$= \frac{\text{adjacent side (AM)}}{\text{hypotenuse (AP)}}.$$

We may find the approximate value of the cosine of any angle by measurement and computation.

Here $\cos 38 = \frac{BN}{BQ} = \frac{2 \cdot 2}{2 \cdot 8}$ approx. $= 0 \cdot 79$ approx.

Note that $\angle BQN = 90° - 38° = 52°$; $\therefore \frac{QN}{QB} = \cos 52°$.

EXERCISE XIX.

(For Tables, see p. 52.)

1. Find by drawing and measurement approximate values of cos 20°, cos 40°, cos 60°.

2. Find by drawing and measurement the angles whose cosines are 0·92, 0·80, 0·31.

3. In Fig. 56, BQ=6 cm., $\angle B = 52°$, find BN.

4. In Fig. 56, BQ=7 cm., $\angle BQN = 37°$, find QN.

5. In Fig. 56, BQ=5 cm., $\angle B = 41°$, find QN.

6. In Fig. 56, BQ=7 cm., BN=3·7 cm., find $\angle B$.

7. A ladder 20 feet long stands against a vertical wall and makes an angle of 72° with the ground. What is the distance of the foot of the ladder from the wall ? What angle does the ladder make with the wall. How high up from the ground does the ladder reach ?

8. A ladder 25 feet long stands against a vertical wall and its foot is 7 feet from the wall. What angle does the ladder make with the ground ?

9. A hill slopes upwards at an angle of 22° with the horizontal. What height does a man rise vertically when he walks 25 yards up the hill ?

10. A man starts at O and walks 150 yards in a direction 37° North of East. How far east of O is he? How far north of O is he?

11. The string of a kite is 750 feet long, and makes an angle of 62° with the ground. How high is the kite?

12. The pole of a bell tent is 8·2 feet long and a slant line of the face of the tent is 10 feet long. What angle does the side of the tent make with the ground?

13. A door is 3 ft. 6 in. wide, and is opened to an angle of 58°, what is the widest packing case that can be pushed in?

14. P is $3\frac{1}{2}$ miles east of Q; R is north of P and $4\frac{1}{2}$ miles from Q. What is the bearing of R from Q?

15. C is 8·5 miles east of A. B is north of A and west 32° north of C. What is the distance of B from C?

16. If the ground rises at a constant angle of 7° with the horizontal, how far must a man walk to rise 100 feet vertically?

17. A picture 45 inches high rests with its lower edge against a wall and makes an angle of 7° with it. What is the distance of the top of the picture from the wall?

18. A man on the top of a cliff 170 feet high observes that the angle of depression of a boat is 27°. How far is the boat from the man?

19. A pendulum 3 ft. long swings backwards and forwards through an angle of 12° each side of the vertical. How high does the lower end of the pendulum rise?

20. A man walks 100 yards up a slope of 20° and then 150 yards up a slope of 15°. What is the total vertical height he has risen?

For additional examples, see Exercise XVII.

Cosines and Tangents.

(Three figure Tables.)

Cosine.	Angle.	Tangent.	Cosine.	Angle.	Tangent.	Cosine.	Angle.	Tangent.
1·000	0°	0	·866	30°	·577	·500	60°	1·732
1·000	1	·017	·857	31	·601	·485	61	1·804
·999	2	·035	·848	32	·625	·470	62	1·881
·999	3	·052	·839	33	·649	·454	63	1·963
·998	4	·070	·829	34	·674	·438	64	2·050
·996	5	·087	·819	35	·700	·423	65	2·144
·994	6	·105	·809	36	·726	·407	66	2·246
·992	7	·123	·799	37	·754	·391	67	2·356
·990	8	·140	·788	38	·781	·375	68	2·475
·988	9	·158	·777	39	·810	·358	69	2·605
·985	10	·176	·766	40	·839	·342	70	2·747
·982	11	·194	·755	41	·869	·326	71	2·904
·978	12	·213	·743	42	·900	·309	72	3·078
·974	13	·231	·731	43	·932	·292	73	3·271
·970	14	·249	·719	44	·966	·276	74	3·487
·966	15	·268	·707	45	1·000	·259	75	3·732
·961	16	·287	·695	46	1·036	·242	76	4·011
·956	17	·306	·682	47	1·072	·225	77	4·331
·951	18	·325	·669	48	1·111	·208	78	4·705
·946	19	·344	·656	49	1·150	·191	79	5·145
·940	20	·364	·643	50	1·192	·174	80	5·671
·934	21	·384	·629	51	1·235	·156	81	6·314
·927	22	·404	·616	52	1·280	·139	82	7·115
·920	23	·424	·602	53	1·327	·122	83	8·144
·913	24	·445	·588	54	1·376	·104	84	9·514
·906	25	·466	·574	55	1·428	·087	85	11·43
·899	26	·488	·559	56	1·483	·070	86	14·30
·891	27	·510	·545	57	1·540	·052	87	19·08
·883	28	·532	·530	58	1·600	·035	88	28·64
·875	29	·554	·515	59	1·664	·017	89	57·29
·866	30	·577	·500	60	1·732	0	90	∞

STAGE C.

DEDUCTIVE DEVELOPMENT.

SECTION I.

CONGRUENCE AND PARALLELISM.

Notes.

1. When learning propositions students should be told not to use the figure printed in the book, but to draw their own figures, and work through the argument on these : a freehand figure is good enough.

2. An excellent procedure in revision is to take a given theorem and trace out all the previous theorems required in its proof. It will be found that in general we are driven back to the fundamental parallel theorems or the fundamental congruence theorems or (more often) to both. If this method is used, the revision of a single theorem becomes the revision of a whole group of theorems and provides excellent material for *viva-voce* work.

A similar method may be applied to Constructions.

Definition.

If two sides of a triangle are equal, the triangle is called **isosceles** ; the angle between the equal sides is often called the **vertical angle**, and the third side is called the base.

The tests for congruence may be used to give a formal proof of the chief property of an isosceles triangle, which might also be deduced from the symmetry of the figure.

THEOREM 11.

(1) If two sides of a triangle are equal, then the angles opposite to those sides are equal.

(2) If two angles of a triangle are equal, then the sides opposite those angles are equal.

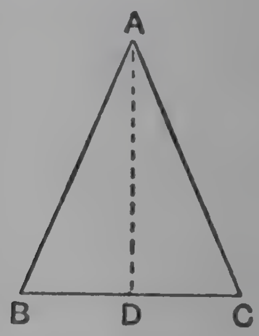


FIG. 57.

ABC is a triangle: let the line bisecting the angle BAC meet BC at D.

(1) *Given* AB = AC.

To prove $\angle$ACB = $\angle$ABC.

In the $\triangle$s ABD, ACD,

AB = AC, given.

AD is common.

$\angle$BAD = $\angle$CAD, constr.

$\therefore$ the $\triangle$s are congruent (2 sides, inc. angle).

$\therefore$ $\angle$ABD = $\angle$ACD.

(2) *Given* $\angle$ABC = $\angle$ACB.

To prove AC = AB.

In the $\triangle$s ABD, ACD,

$\angle$ABD = $\angle$ACD, given.

$\angle$BAD = $\angle$CAD, constr.

AD is common.

$\therefore$ the $\triangle$s are congruent (2 angles, corr. side).

$\therefore$ AB = AC.

Q.E.D.

THEOREM 12.

Two right-angled triangles are congruent if the hypotenuse and a side of one are respectively equal to the hypotenuse and a side of the other.

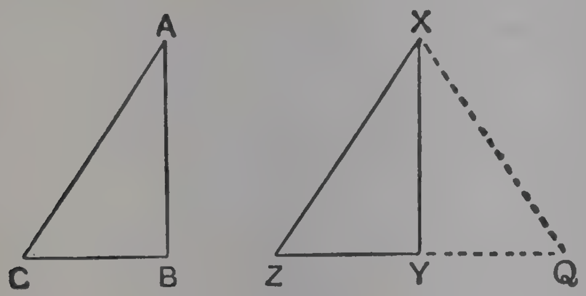


FIG. 58.

Given $\angle ABC = 90° = \angle XYZ$, $AC = XZ$, $AB = XY$.

To prove $\triangle ABC \equiv \triangle XYZ$.

Produce ZY to Q, making $YQ = BC$, join XQ.

Since $\angle ZYX = 90°$ and ZYQ is a straight line, $\angle QYX = 90°$.

$\therefore$ in the $\triangle$s ABC, XYQ,

$AB = XY$, given.

$BC = YQ$, constr.

$\angle ABC = \angle XYQ$, right angles.

$\therefore$ $\triangle ABC \equiv \triangle XYQ$ (2 sides, inc. angle).

$\therefore$ $AC = XQ$ and $\angle C = \angle Q$.

But $AC = XZ$, given. $\therefore$ $XZ = XQ$ and $\triangle XZQ$ is isosceles.

$\therefore$ $\angle Q = \angle Z$.

But $\angle C = \angle Q$. $\therefore$ $\angle C = \angle Z$.

$\therefore$ in the $\triangle$s ABC, XYZ,

$AB = XY$, given.

$\angle ABC = \angle XYZ$, right angles.

$\angle C = \angle Z$, proved.

$\therefore$ $\triangle ABC \equiv \triangle XYZ$ (2 angles, corr. side).

Q.E.D.

$1\frac{1}{2}$ 25

EXERCISE XX.

1. The vertical angle of an isosceles triangle is 110° ; what are the base angles ?

2. One base angle of an isosceles triangle is 62° ; what is the vertical angle ?

3. Find the angles of an isosceles triangle if (i) the vertical angle is double a base angle, (ii) a base angle is double the vertical angle.

4. In the triangle ABC, $\angle BAC = 2\angle ABC$ and $\angle ACB - \angle ABC = 36°$; prove that the triangle is isosceles.

5. A, B, C are three points on a circle, centre O ; $\angle AOB = 100°$, $\angle BOC = 140°$; calculate the angles of the triangle ABC.

6. In Fig. 59, if AB=AC, find x in terms of y.

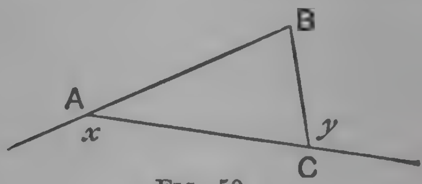


FIG. 59.

7. D is a point on the base BC of the isosceles triangle ABC such that BD=BA ; if $\angle BAD = x°$ and $\angle DAC = y°$, express x in terms of y.

8. ABCDE is a regular pentagon, prove that the line bisecting the angle BAC is perpendicular to AE.

9. In the triangle ABC, AB=AC ; D is a point in AC such that AD=BD=BC ; calculate $\angle BAC$.

10. ABC is an equilateral triangle ; BC is produced to D so that BC=CD ; prove that $\angle BAD = 90°$.

11. In the $\triangle ABC$, AB=AC ; AB is produced to D so that BD=BC ; prove that $\angle ACD = 3\angle ADC$.

12. P is a point on the line bisecting $\angle BAC$; through P a line is drawn parallel to AC and cutting AB at Q ; prove AQ=QP.

13. In $\triangle ABC$, AB=AC ; D is a point on AC produced such that BD=BA ; if $\angle CBD = 36°$, prove BC=CD.

14 If in Fig. 60, AB=AC and CP=CQ, prove

$$\angle SRP = 3\angle RPC.$$

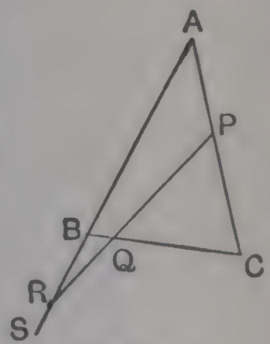


FIG. 60.

15. The base BC of the isosceles triangle ABC is produced to D ; the lines bisecting $\angle ABC$ and $\angle ACB$ meet at I ; prove $\angle ACD = \angle BIC$.

16. In the quadrilateral ABCD, AB=AD and $\angle$ABC=$\angle$ADC; prove CB=CD.

17. ABC is an acute-angled triangle; AB<AC; the circle, centre A, radius AB cuts BC at D; prove that $\angle$ABC+$\angle$ADC=180°.

18. A, B, C are three points on a circle, centre O; prove $\angle$ABC =$\angle$OAB+$\angle$OCB, if O lies between BA and BC.

19. AB, AC are two chords of a circle, centre O; if $\angle$BAC=90°, prove that BOC is a straight line.

20. In the $\triangle$ABC, AB=AC; the bisectors of the angles ABC and ACB meet at I; prove that IB=IC.

21. AD is an altitude of the equilateral triangle ABC; ADX is another equilateral triangle, prove that DX is perpendicular to AB or AC.

22. BC is the base of an isosceles triangle ABC; P, Q are points on AB, AC such that AP=PQ=QB=BC; calculate $\angle$BAC.

23. D is the mid-point of the base BC of the triangle ABC; if AD=DB, prove $\angle$BAC=90°.

24. In the quadrilateral ABCD, AB=CD and $\angle$ABC=$\angle$DCB, prove $\angle$BAD=$\angle$CDA.

25. In the $\triangle$ABC, AB>AC; D is a point on AB such that AD=AC; prove $\angle$ABC+$\angle$ACB=2$\angle$ADC.

26. The triangle ABC is right-angled at A; AD is the perpendicular from A to BC; P is a point on CB such that CP=CA; prove AP bisects $\angle$BAD

27. The vertical angles of two isosceles triangles are supplementary; prove that their base angles are complementary.

28. Draw two triangles ABC, XYZ which are such that AB=XY, AC=XZ, $\angle$ABC=$\angle$XYZ but are not congruent. Prove

$$\angle ACB+\angle XZY=180°.$$

29. In the $\triangle$ABC, AB=AC; P is any point on BC produced. PX, PY are the perpendiculars from P to AB, AC produced; prove $\angle$XPB=$\angle$YPB.

30. OA=OB=OC and $\angle$BAC is acute; prove $\angle$BOC=2$\angle$BAC.

31. In the $\triangle$ABC, AB=AC; AB is produced to D; prove $\angle$ACD−$\angle$ADC=2$\angle$BCD.

THEOREM 13.

Inequalities.

(1) If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

(2) If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

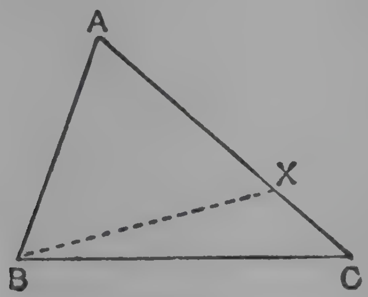


FIG. 61.

(1) *Given* $AC > AB$.

To prove $\angle ABC > \angle ACB$.

From AC cut off a part AX equal to AB. Join BX.

Since $AB = AX$, $\angle ABX = \angle AXB$.

But ext. $\angle AXB >$ int. opp. $\angle XCB$,

$\therefore \angle ABX > \angle XCB$.

But $\angle ABC > \angle ABX$,

$\therefore \angle ABC > \angle XCB$ or $\angle ACB$.

(2) *Given* $\angle ABC > \angle ACB$.

To prove $AC > AB$.

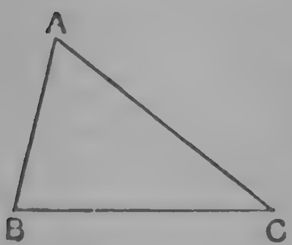


FIG. 62.

If AC is not greater than AB, it must either be equal to AB, or less than AB.

If $AC = AB$, $\angle ABC = \angle ACB$,

which is contrary to hypothesis.

If $AC < AB$, $\angle ABC < \angle ACB$,

which is contrary to hypothesis.

$\therefore$ AC must be greater than AB.

Q.E.D.

THEOREM 14.

Of all straight lines that can be drawn to a given straight line from an external point, the perpendicular is the shortest.

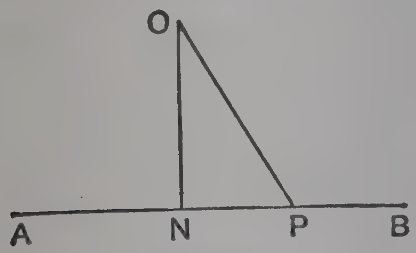


FIG. 63.

Given a fixed point O and a fixed line AB.

ON is the perpendicular from O to AB, and OP is any other line from O to AB.

To prove ON$<$OP.

Since the sum of the angles of a triangle is 2 rt. angles, and since $\angle$ONP$=1$ rt. angle.

$\therefore$ $\angle$NPO $+$ $\angle$NOP $=1$ rt. angle.

$\therefore$ $\angle$NPO<1 rt. angle.

$\therefore$ $\angle$NPO$<$ $\angle$ONP.

$\therefore$ ON$<$OP. Q.E.D.

Distance between two Points.

By the distance between two points, we mean the shortest distance they are apart, and we take this as measured by the straight line which joins the two points.

If ABC is any triangle, so that C does not lie on AB, the distance of A from C + the distance of C from B is greater than the distance of A from B. This fact is often stated in the following way.

THEOREM 15.

Any two sides of a triangle are together greater than the third side

(A formal proof is given in Appendix I.)

EXERCISE XXI.

1. The bisectors of the angles ABC, ACB of $\triangle$ ABC meet at I; if AB>AC, prove that IB>IC.

2. AD is a median of $\triangle$ABC; if BC<2AD, prove that $\angle$BAC <90°.

3. ABC is an equilateral triangle; P is any point on BC; prove AP>BP.

4. In $\triangle$ABC, the bisector of $\angle$BAC cuts BC at D; prove BA>BD.

5. AD is a median of $\triangle$ABC; if AB>AC, prove that $\angle$BAD <$\angle$CAD.

6. In $\triangle$ABC, AB=AC; BC is produced to any point D; P is any point on AB; DP cuts AC at Q; prove AQ>AP.

7. ABC is a triangle; the bisector of $\angle$BAC cuts BC at D; if AB>AC, prove BD>DC.

8. ABC is an acute-angled triangle, such that $\angle$ABC=2$\angle$ACB; prove AC<2AB.

9. ABCD is a quadrilateral; prove that AB+BC+CD>AD.

10. Prove that any side of a triangle is less than half its perimeter.

11. How many triangles can be drawn such that two of the sides are of lengths 4 feet, 7 feet, and such that the third side contains a whole number of feet?

12. ABC is a $\triangle$; D is any point on BC; prove that AD<$\frac{1}{2}$(AB +BC+CA).

13. ABCD is a quadrilateral; AB<BC; $\angle$BAD<$\angle$BCD; prove AD>CD.

14. O is any point inside the triangle ABC; prove that (i) $\angle$BOC >$\angle$BAC; (ii) BO+OC<BA+AC.

15. A, B, are any two points on the same side of CD, A′ is the image of A in CD (*i.e.*, CD bisects AA′ at right angles); A′B cuts CD at O; P is any other point on CD; prove that AP+PB>AO+OB.

16. AD is a median of $\triangle$ABC; prove AD<$\frac{1}{2}$(AB+AC).

17. O is any point inside $\triangle$ABC; prove OA+OB+OC>$\frac{1}{2}$(BC +CA+AB).

18. Prove that the sum of the diagonals of a quadrilateral is greater than the semiperimeter and less than the perimeter of the quadrilateral.

Definition. A quadrilateral whose opposite sides are parallel is called a parallelogram (p. 28).

THEOREM 16.

(1) The opposite sides and angles of a parallelogram are equal.

(2) Each diagonal bisects the parallelogram.

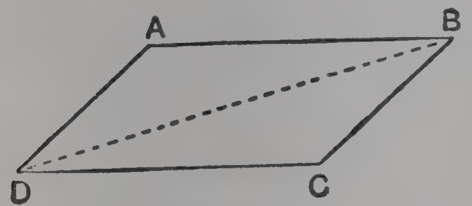


FIG. 64.

Given ABCD is a parallelogram.

To prove (1) $AB = CD$ and $AD = BC$.

$\angle DAB = \angle DCB$ and $\angle ABC = \angle ADC$.

(2) AC and BD each bisect the parallelogram.

Join BD.

In the $\triangle$s ADB, CBD,

$\angle ADB = \angle CBD$, alt. $\angle$s., $AD \parallel BC$.

$\angle ABD = \angle CDB$, alt. $\angle$s., $AB \parallel DC$.

BD is common.

$\therefore$ $\triangle ADB \equiv \triangle CBD$ (2 angles, corr. side).

$\therefore$ $AB = CD$, $AD = BC$, $\angle DAB = \angle BCD$,

and BD bisects the parallelogram.

Similarly, by joining AC it may be proved that $\angle ABC = \angle ADC$, and that AC bisects the parallelogram.

Q.E.D.

THEOREM 17.

The diagonals of a parallelogram bisect one another.

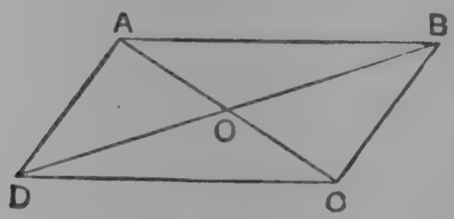


FIG. 65.

The diagonals AC, BD of the parallelogram ABCD intersect at O.

To prove AO = OC and BO = OD.

In the $\triangle$s AOD, COB,

$\angle$DAO = $\angle$BCO, alt. $\angle$s., AD || BC.

$\angle$ADO = $\angle$CBO, alt. $\angle$s., AD || BC.

AD = BC, opp. sides of || gram.

$\therefore$ $\angle$AOD $\equiv$ $\triangle$COB (2 angles, corr. side).

$\therefore$ AO = CO and BO = DO.

Q.E.D.

THEOREM 18.

The straight lines which join the ends of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

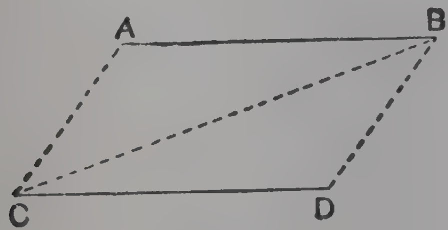


FIG. 66.

Given AB is equal and parallel to CD.

To prove AC is equal and parallel to BD.

Join BC.

In the $\triangle$s ABC, DCB,

$$AB = DC, \text{ given.}$$

$$BC \text{ is common.}$$

$$\angle ABC = \angle DCB \text{ alt. angles, } AB \text{ being } \parallel \text{ to } CD.$$

$$\therefore \triangle ABC \equiv \triangle DCB \text{ (2 sides, inc. angle).}$$

$$\therefore AC = DB \text{ and } \angle ACB = \angle DBC.$$

But these are alt. angles. $\therefore$ AC is parallel to DB.

Q.E.D.

This theorem can also be stated as follows :

A quadrilateral which has one pair of equal and parallel sides is a parallelogram.

THEOREM 19.

A quadrilateral is a parallelogram if
either (1) its opposite angles are equal,
or (2) its opposite sides are equal,
or (3) its diagonals bisect each other.

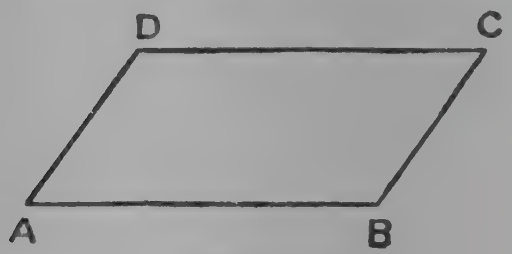


FIG. 67.

(1) *Given* $\angle A = \angle C$ and $\angle B = \angle D$.
To prove ABCD is a parallelogram,
The angles of any quadrilateral add up to 360°.

$$\therefore \angle A + \angle B + \angle C + \angle D = 360°.$$

But $\angle A = \angle C$ and $\angle B = \angle D$, given.

$$\therefore 2\angle C + 2\angle D = 360°.$$
$$\therefore \angle C + \angle D = 180°.$$

$\therefore$ AD is parallel to BC.
Similarly, it may be proved that AB is parallel to DC.
$\therefore$ ABCD is a parallelogram. Q.E.D.

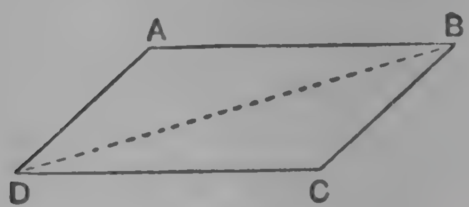


FIG. 68.

(2) *Given* AB = CD and AD = CB.
To prove ABCD is a parallelogram.
Join BD.
In the $\triangle$s ADB, CBD AD = CB, given.
AB = CD, given.
DB is common.

$\therefore$ $\triangle ADB \equiv \triangle CBD$ (3 sides).

$\therefore$ $\angle ADB = \angle CBD$, but these are alternate angles,

$\therefore$ AD is parallel to BC.

Similarly, since $\angle ABD = \angle CDB$, AB is parallel to DC.

$\therefore$ ABCD is a parallelogram. Q.E.D.

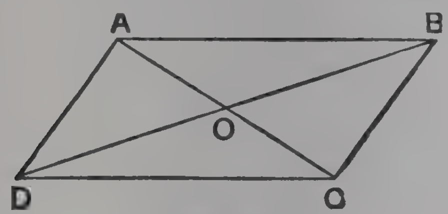


FIG. 69.

Given AO = OC and BO = OD.

To prove ABCD is a parallelogram.

In the $\triangle$s AOB, COD, AO = OC, given.

BO = OD, given.

$\angle AOB = \angle COD$, vert. opp.

$\therefore$ $\triangle AOB \equiv \triangle COD$ (2 sides, inc. angle).

$\therefore$ $\angle BAO = \angle DCO$, but these are alternate angles.

$\therefore$ AB is parallel to CD.

Similarly, it may be proved that AD is parallel to BC.

$\therefore$ ABCD is a parallelogram. Q.E.D.

Definitions.

A **rectangle** is a parallelogram, *one* angle of which is a right angle.

A **square** is a rectangle, having two adjacent sides equal.

A **rhombus** is a parallelogram, having two adjacent sides equal, but none of its angles right angles.

A **trapezium** is a quadrilateral having one pair of opposite sides parallel.

EXERCISE XXII.

1. Prove that all the sides of a rhombus are equal.

2. Prove that the diagonals of a rectangle are equal.

3. Prove that the diagonals of a rhombus intersect at right angles.

4. Prove that the diagonals of a square are equal and cut at right angles.

5. The diagonals of the rectangle ABCD meet at O; $\angle$BOC$=44°$; calculate $\angle$OAD.

6. ABCD is a rectangle; $\angle$BAC$=32°$; calculate $\angle$DBC.

7. ABCD is a rhombus; $\angle$ABC$=56°$; calculate $\angle$ACD.

8. ABCD is a parallelogram; prove that B and D are equidistant from AC.

9. ABCD is a parallelogram whose diagonals intersect at O; prove that $\angle$DAC$+\angle$DBC$=\angle$DOC.

10. The diagonals of the parallelogram ABCD cut at O; any line through O cuts AB, CD at X, Y; prove XO=OY.

11. Two straight lines POQ, ROS cut at O; if PQ=RS and PR=QS, prove $\angle$RPO$=\angle$QSO.

12. In the quadrilateral ABCD, AB=CD and AC=BD; prove that AD is parallel to BC.

13. E is a point inside the square ABCD; a square AEFG is described on the same side of AE as D; prove BE=DG.

14. ABC is any triangle; BY, CZ are lines parallel to AC, AB cutting a line through A parallel to BC in Y, Z; prove AY=AZ.

15. ABCD is a parallelogram; P is the mid-point of BC; DP and AB are produced to meet at Q; prove AQ=2AB.

16. ABCD, ABXY are two parallelograms; BC and BX are different lines; prove that DCXY is a parallelogram.

17. ABCD is a quadrilateral with BC parallel to AD. If AC, DB bisect the angles $\angle$BAD, $\angle$CDA, prove that AB=BC=CD.

18. The diagonals of a square ABCD cut at O; from AB a part AK is cut off equal to AO; prove $\angle$AOK$=3\angle$BOK.

19. ABCD is a straight line such that AB=BC=CD; BCPQ is a rhombus; prove that AQ is perpendicular to DP.

20. ABCD is a parallelogram; the bisector of $\angle$ABC cuts AD at X; the bisector of $\angle$BAD cuts BC at Y; prove XY=CD.

21. ABCD is a parallelogram such that the bisectors of $\angle$s DAB, ABC meet on CD; prove AB=2BC.

22. In $\triangle$ABC, $\angle$BAC$=90°$; BADH, ACKE are squares outside the triangle; prove that HAK is a straight line.

23. The diagonals of the rectangle ABCD cut at O; AO$>$AB; the circle, centre A, radius AO cuts AB produced at E; if $\angle$AOB $=4\angle$BOE, calculate $\angle$BAC.

HARDER EXAMPLES ON CONGRUENCE AND ISOSCELES TRIANGLES.

EXERCISE XXIII.

1. If the diagonal AC of the quadrilateral ABCD bisects the angles DAB, DCB, prove that AC bisects BD at right angles.

2. ABCD is a quadrilateral; E, F are the mid-points of AB, CD; if $\angle AEF = 90° = \angle EFD$, prove that $AD = BC$.

3. Two lines POQ, ROS bisect each other, prove that the triangles PRS, PQS are equal in area.

4. Two lines POQ, ROS intersect at O; SP and QR are produced to meet at T; if $OP = OR$ and $OS = OQ$, prove $TS = TQ$.

5. ABC is any triangle; ABX, ACY are equilateral triangles external to ABC; prove $CX = BY$.

6. Two unequal circles, centres A, B intersect at X, Y; prove that AB bisects XY at right angles.

7. D is a point on the side AB of $\triangle ABC$, such that $AD = DC = CB$; AC is produced to E; prove $\angle ECB = 3\angle ACD$.

8. In the $\triangle ABC$, $\angle BAC$ is obtuse; the perpendicular bisectors of AB, AC cut BC at X, Y; prove $\angle XAY = 2\angle BAC - 180°$.

9. In the $\triangle ABC$, $AB = AC$ and $\angle BAC > 60°$; the perpendicular bisector of AC meets BC at P; prove $\angle APB = 2\angle ABP$.

10. D is the mid-point of the side AB of $\triangle ABC$; the bisector of $\angle ABC$ cuts the line through D parallel to BC at K; prove

$$\angle BKA = 90°.$$

11. In the $\triangle ABC$, $\angle BAC = 90°$, and $AB = AC$; P, Q are points on AB, AC such that $AP = AQ$; prove that the perpendicular from A to BQ bisects CP.

12. X, Y are the mid-points of the sides AB, AC of the $\triangle ABC$; P is any point on a line through A parallel to BC; PX, PY are produced to meet BC at Q, R; prove $QR = BC$.

13. ABC is a triangle; the perpendicular bisectors of AB, AC meet at O; prove $OB = OC$.

14. ABC is a triangle; the lines bisecting the angles ABC, ACB meet at I; prove that the perpendiculars from I to AB, AC are equal.

15. The sides AB, AC of the triangle ABC are produced to H, K; the lines bisecting the angles HBC, KCB meet at I; prove that the perpendiculars from I to AH, AK are equal.

16. Two circles have the same centre; a straight line PQRS cuts one circle at P, S and the other at Q, R; prove PQ=RS.

17. The line joining the mid-points E, F of AB, AC is produced to G so that EF=FG; prove that BE is equal and parallel to CG.

18. In the 5-sided figure ABCDE, the angles at A, B, C, D are each 120°; prove that AB+BC=DE.

19. ABC is a triangle; lines are drawn through C parallel to the bisectors of the angles CAB, CBA to meet AB produced in D, E; prove that DE equals the perimeter of the triangle ABC.

20. AB, BC, CD are chords of a circle, centre O; if $\angle$AOB=108°, $\angle$BOC=60°, $\angle$COD=36°, prove AB=BC+CD. (From BA cut off BQ equal to BO: join OQ.)

21. In the triangles ABC, XYZ, if BC=YZ, $\angle$ABC=$\angle$XYZ, AB+AC=XY; prove $\angle$BAC=2$\angle$YXZ.

22. ABC is an equilateral triangle; a line parallel to AC cuts BA, BC at P, Q; AC is produced to R so that BQ=CR; prove that PR bisects CQ.

23. In $\triangle$ABC, $\angle$BAC=90°; ABPQ, ACRS, BCXY are squares outside ABC; prove that (i) BQ is parallel to CS; (ii) BR is perpendicular to AX.

24. In $\triangle$ABC, $\angle$BAC=90°; BCPQ, ACHK are squares outside ABC; AC cuts PH at D; prove AB=2CD and PD=DH.

25. In $\triangle$ABC, AB=AC; P is any point on BC; PX, PY are the perpendiculars from P to AB, AC; CD is the perpendicular from C to AB; prove PX+PY=CD.

Formal Constructions.

In the following constructions the instruments allowed are restricted to

(i) A straight edge (ungraduated ruler) for (i) joining two given points by a straight line, and (ii) producing a given straight line.

(ii) Compasses for (i) drawing a circle of given centre and given radius, and (ii) cutting off from a straight line a length equal to that of a given straight line.

Formal proofs of these constructions are given.

CONSTRUCTION 1.

From a given point in a given straight line, draw a straight line making with the given line an angle equal to a given angle.

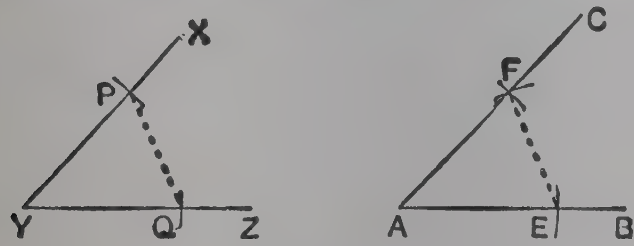


FIG. 70.

Given a point A on a given line AB and an angle XYZ.

To construct a line AC such that $\angle CAB = \angle XYZ$.

With centre Y and any radius, draw an arc of a circle cutting YX, YZ at P, Q.

With centre A and the same radius, draw an arc of a circle EF, cutting AB at E.

With centre E and radius equal to QP, describe an arc of a circle, cutting the arc EF at F.

Join AF and produce it to C.

Then AC is the required line.

Proof. Join PQ, EF.

In the $\triangle$s PYQ, FAE,

$$YP = AF, \text{ constr.}$$

$$YQ = AE, \text{ constr.}$$

$$PQ = EF, \text{ constr.}$$

$$\therefore \triangle PYQ \equiv \triangle FAE \text{ (3 sides).}$$

$$\therefore \angle XYZ = \angle BAC.$$

Q.E.F.

CONSTRUCTION 2.

Bisect a given angle.

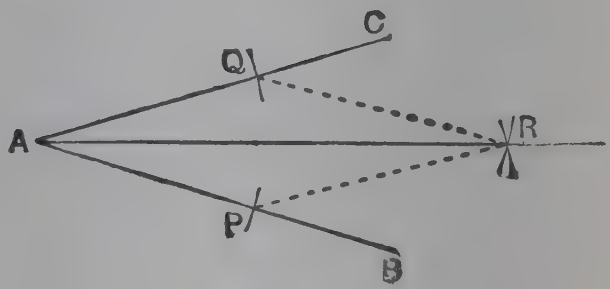


FIG. 71.

Given an angle BAC.

To construct a line bisecting the angle.

With A as centre and any radius, draw an arc of a circle, cutting AB, AC at P, Q.

With centres P, Q and with any sufficient radius, the same for each, draw arcs of circles, cutting at R. Join AR.

Then AR is the required bisector.

Proof. Join PR, QR.

In the $\triangle$s APR, AQR,

AP = AQ, radii of the same circle.

PR = QR, radii of equal circles.

AR is common.

$\therefore$ $\triangle APR \equiv \triangle AQR$ (3 sides).

$\therefore$ $\angle PAR = \angle QAR$.

Q.E.F.

CONSTRUCTION 3.

Draw the perpendicular bisector of a given finite straight line.

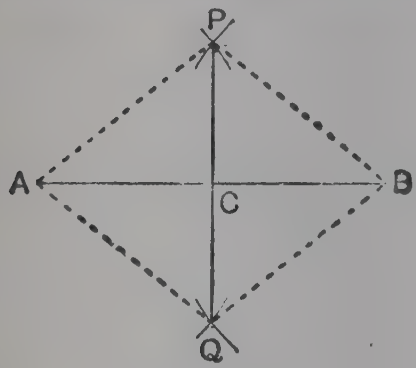


FIG. 72.

Given a finite line AB.

To construct the line bisecting AB at right angles.

With centres A, B and any sufficient radius, the same for each, draw arcs of circles to cut at P, Q.

Join PQ and let it cut AB at C.

Then C is the mid-point of AB, and PCQ bisects AB at right angles.

Proof. Join PA, PB, QA, QB.

In the $\triangle$s PAQ, PBQ,

PA = PB, radii of equal circles.
QA = QB, radii of equal circles.
PQ is common.
$\therefore$ $\triangle$PAQ $\equiv$ $\triangle$PBQ (3 sides).
$\therefore$ $\angle$APQ = $\angle$BPQ.

In the $\triangle$s APC, BPC,

PA = PB, radii of equal circles.
PC is common.
$\angle$APC = $\angle$BPC, proved.
$\therefore$ $\triangle$APC $\equiv$ $\triangle$BPC (2 sides, inc. angle).
$\therefore$ AC = CB.
and $\angle$ACP = $\angle$BCP.

But these are adjacent angles, $\therefore$ each is a right angle.

Q.E.F.

CONSTRUCTION 4.

Draw a straight line at right angles to a given straight line from a given point in it.

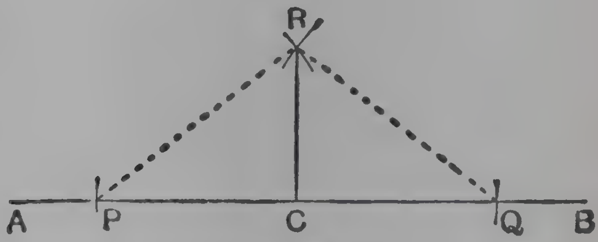


FIG. 73.

Given a point C on a line AB.

To construct a line from C perpendicular to AB.

With centre C and any radius, draw an arc of a circle cutting AB at P, Q.

With centres P, Q and any sufficient radius, the same for each, draw arcs of circles to cut at R. Join CR.

Then CR is the required perpendicular.

Proof. Join PR, QR.

In the $\triangle$s RCP, RCQ,

RP = RQ, radii of equal circles.

CP = CQ, radii of the same circle.

CR is common.

$\therefore$ $\triangle RCP \equiv \triangle RCQ$ (3 sides).

$\therefore$ $\angle RCP = \angle RCQ$.

But these are adjacent angles, $\therefore$ each is a right angle.

Q.E.F.

CONSTRUCTION 5.

Draw a perpendicular to a given straight line of unlimited length from a given point outside it.

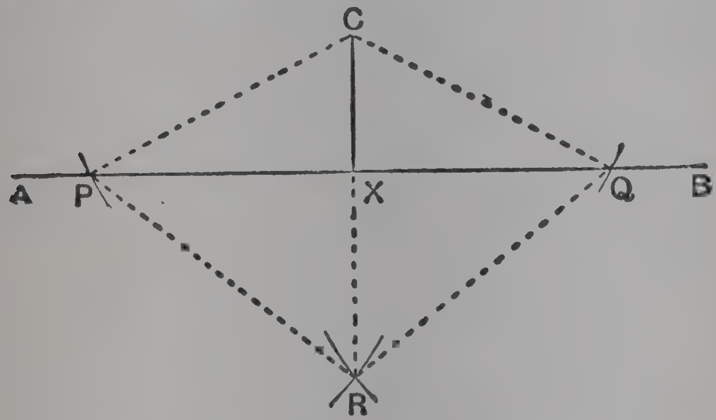


FIG. 74.

Given a line AB and a point C outside it.

To construct a line from C perpendicular to AB.

With C as centre and any sufficient radius, draw an arc of a circle, cutting AB at P, Q.

With P, Q as centres and any sufficient radius, the same for each, draw arcs of circles, cutting at R. Join CR and let it cut AB at X.

Then CX is perpendicular to AB.

Proof. Join, CP, CQ, RP, RQ.

In the $\triangle$s CPR, CQR,

CP = CQ, radii of the same circle.

RP = RQ, radii of equal circles.

CR is common.

$\therefore$ $\triangle$CPR $\equiv$ $\triangle$CQR (3 sides). $\angle$PCR = $\angle$QCR.

In the $\triangle$s CPX, CQX,

CP = CQ, radii.

CX is common.

$\angle$PCX = $\angle$QCX, proved.

$\therefore$ $\triangle$CPX $\equiv$ $\triangle$CQX (2 sides, inc. angle). $\therefore$ $\angle$CXP = $\angle$CXQ.

But these are adjacent angles, $\therefore$ each is a right angle.

Q.E.F.

CONSTRUCTION 6.

Through a given point, draw a straight line parallel to a given straight line.

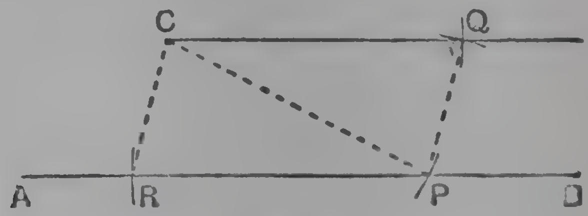


FIG. 75.

Given a line AB and a point C outside it.

To construct a line through C parallel to AB.

With C as centre and any sufficient radius, draw an arc of a circle PQ, cutting AB at P.

With P as centre and the same radius, draw an arc of a circle, cutting AB at R.

With centre P and radius equal to CR, draw an arc of a circle, cutting the arc PQ at Q on the same side of AB as C. Join CQ.

Then CQ is parallel to AB.

Proof. Join CR, CP, PQ.

In the $\triangle$s CRP, PQC

$$CR = PQ, \text{ constr.}$$

$$RP = QC \text{ radii of equal circles.}$$

$$PC \text{ is common.}$$

$$\therefore \triangle CRP \equiv \triangle PQC \text{ (3 sides).}$$

$$\therefore \angle CPR = \angle PCQ.$$

But these are alternate angles, $\therefore$ CQ is parallel to RP.

Q.E.F.

EXERCISE XXIV.

1. Construct an equilateral triangle, given one side.

2. Draw any angle AOB ; with O as centre and any radius (not too short), describe a circle cutting OA, OB at P, Q ; with P, Q as centres and any radius (not too short), describe two equal circles cutting at R. Measure $\angle$AOR, $\angle$BOR.

3. Draw any straight line AB ; with A, B as centres and any radius (not too short), describe two equal circles cutting at P, Q. Join PQ and let it cut AB at R. Measure AR, RB and $\angle$ARP.

4. Draw any straight line AB and take any point C on it. With C as centre, describe any circle cutting AB at P, Q ; with P, Q as centres and any radius (not too short), describe two equal circles cutting at R. Join CR. Measure $\angle$ACR.

5. Draw any straight line AB and take any point C outside it. With C as centre, describe any circle cutting AB at P, Q ; with P, Q as centres and any radius (not too short), describe two equal circles cutting at R. Join CR and let it cut AB at S. Measure $\angle$ASC.

6. Draw any straight line AB and take any point C outside it. Take any point P on AB. Join CP and bisect it at Q. With Q as centre and QC as radius, describe a circle, cutting AB at R. Join CR. Measure $\angle$ARC.

7. With any point O as centre, describe a circle ; draw any chord PQ : construct the perpendicular bisector of PQ. Does it pass through O ?

8. Draw a triangle ABC (not isosceles) ; construct the perpendicular bisectors of AB and AC ; let them meet at O ; with O as centre and OA as radius, describe a circle. Does the circle pass through B and C ?

9. In Fig. 76, without producing AB, construct a line through C perpendicular to AB.

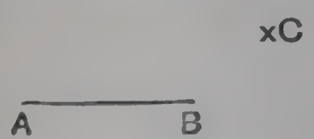


FIG. 76.

10. Draw a line AB, construct a line through B perpendicular to AB *without* producing AB.

11. Draw an obtuse-angled triangle ABC ; construct the perpendiculars from each vertex to the opposite side. Are they concurrent ?

12. Draw a circle of radius 3 cm. and take points A, B, C on it such that AB=4 cm., AC=5 cm. Measure ∠BAC: is there more than one answer?

13. Draw a line AB and take any two points C, D outside it; construct a point P on AB such that PC=PD.

14. Draw any triangle (not isosceles) and construct the bisectors of its three angles. What do you notice about them?

15. Construct (without using a protractor) angles of (i) 30°, (ii) 45°, (iii) 105°, (iv) 255°.

16. Draw an obtuse angle and construct lines dividing it into four equal angles.

17. Draw a triangle ABC (not isosceles); construct a point P on BC such that the perpendiculars from P to AB and AC are equal.

18. Draw a right angle and construct the lines trisecting it.

19. Draw a line PQ (see Fig. 77), cutting two other lines AB, CD at P, Q; the bisectors of ∠s APQ, CQP meet at H; the bisectors

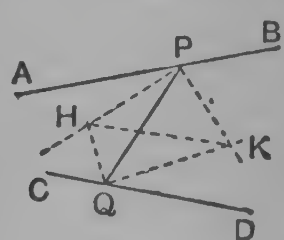


FIG. 77.

of ∠s BPQ, DQP meet at K; verify that HK when produced passes through the point of intersection of AB and CD and bisects the angle between them.

Construction of Triangles and Quadrilaterals from sufficient data.

It is suggested that Constructions 7, 8, 9 should be omitted at a first reading. These are *not* required for Ex. 1-27 in Exercise XXV.

CONSTRUCTION 7.

Draw a triangle, given two angles and the perimeter.

Given two angles X, Y and a line HK.

To construct a triangle having two of its angles equal to X and Y and its perimeter equal to HK.

Construct lines PH, QK on the same side of HK such that $\angle PHK = \angle X$ and $\angle QKH = \angle Y$.

Construct lines HA, KA intersecting at A and bisecting the angles PHK, QKH.

Construct through A, lines AB, AC parallel to PH, QK, cutting HK at B, C.

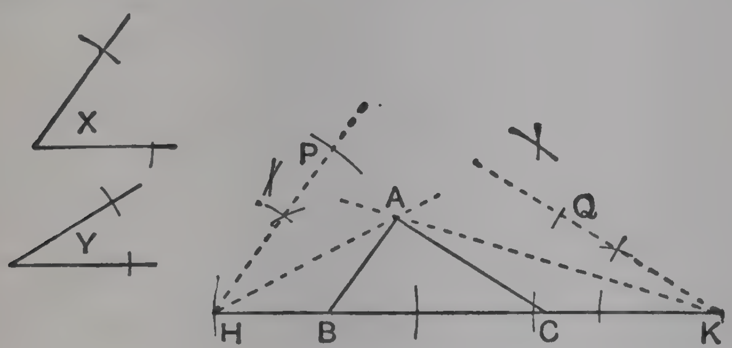


FIG. 78.

Then ABC is the required triangle.

Proof. $\angle BAH = \angle AHP$, since AB is parallel to PH.

$\angle BHA = \angle AHP$, constr.

$\therefore \angle BAH = \angle BHA$.

$\therefore BH = BA$.

Similarly it may be proved that $CK = CA$.

$\therefore AB + BC + CA = HB + BC + CK = HK$.

Also $\angle ABC = \angle PHK = \angle X$, corresp. $\angle$s., $AB \parallel PH$

and $\angle ACB = \angle QKH = \angle Y$, corresp. $\angle$s., $AC \parallel QK$.

$\therefore$ ABC is the required triangle.

Q.E.F.

CONSTRUCTION 8.

Draw a triangle given one angle, the side opposite that angle and the sum of the other two sides.

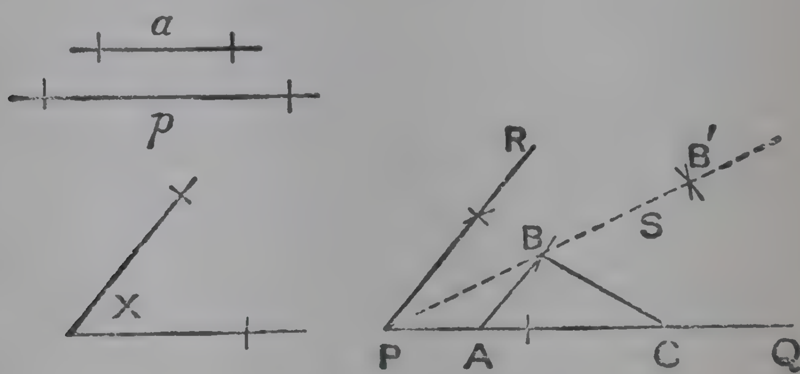


FIG. 79.

Given two lines a, p and an angle X.

To construct a triangle ABC such that $BC = a$, $BA + AC = p$.

$$\angle BAC = \angle X.$$

Draw a line PQ and from it cut off a part PC equal to p.

Construct a line PR such that $\angle RPQ = \angle X$.

Construct the line PS bisecting $\angle RPQ$.

With centre C and radius equal to a, draw an arc of a circle cutting PS at B (or B′). Through B (or B′) draw a line parallel to PR to meet PC at A (or A′). Join AB, BC.

Then ABC is the required triangle.

Proof. $\angle APB = \angle BPR$, constr.

$\angle BPR = \angle PBA$, alt. $\angle$s., $PR \parallel AB$.

$\therefore\ \angle APB = \angle PBA. \quad \therefore\ AP = AB.$

$\therefore\ BA + AC = PA + AC = PC = p.$

Also $\angle BAC = \angle RPC$, corresp. $\angle$s., $AB \parallel PR$

$= \angle X$,

and $CB = a$, constr.

$\therefore$ ABC is the required triangle.

Note.—$\triangle$ A′B′C also satisfies the given conditions. There are therefore two distinct (but congruent) solutions of the problem.

CONSTRUCTION 9.

Given the angle BAC, construct points P, Q on AB, AC such that PQ is of given length and the angle APQ of given size.

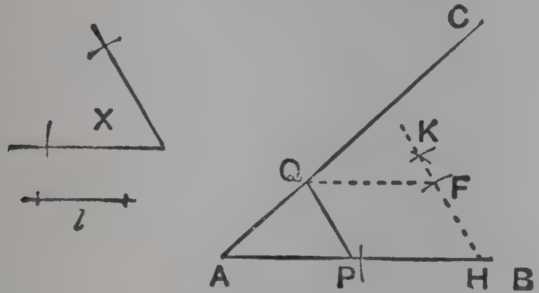


FIG. 80.

Given the angle BAC, a line l and an angle X.

To construct points P, Q on AB, AC such that PQ equals l and $\angle$APQ equals $\angle$X.

Take any point H on AB and construct a line HK such that $\angle AHK = \angle X$.

From HK cut off HF equal to l. Through F draw FQ parallel to AB to cut AC in Q. Through Q draw QP parallel to FH to cut AB in P.

Then PQ is the required line.

Proof. By construction, PQFH is a parallelogram,

$$\therefore \ PQ = HF = l;$$

and $\angle QPA = \angle FHA = \angle X$, corresp. $\angle$s., $PQ \parallel HF$.

$\therefore$ PQ is the required line.

Q.E.F.

CONSTRUCTION 10.

Describe a square on a given straight line.

Given a line AB.

To construct a square on AB.

From A draw a line AC perpendicular to AB; from AC cut off AP equal to AB.

Through P draw PQ parallel to AB.

Through B draw BQ parallel to AP, cutting PQ at Q.

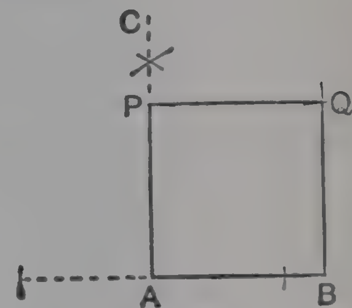


FIG. 81.

Then ABQP is the required square.

Proof. By construction, ABQP is a parallelogram.

But $\angle BAP = 90°$, $\therefore$ ABQP is a rectangle.

But $AB = AP$, $\therefore$ ABQP is a square. Q.E.F.

EXERCISE XXV.

1. Construct, *when possible*, the triangle ABC from the following measurements, choosing your own unit. If there are two different solutions, construct both:

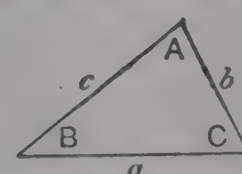


FIG. 82.

(i) $a=3$, $b=4$, $c=5$, measure A.
(ii) $a=3$, $b=4$, $c=8$, measure A.
(iii) $a=5$, $B=30°$, $C=45°$, measure b.
(iv) $a=4$, $A=48°$, $B=33°$, measure b.
(v) $a=7$, $A=110°$, $B=40°$, measure b.
(vi) $a=5$, $B=125°$, $C=70°$, measure b.
(vii) $b=5$, $c=7$, $C=72°$, measure a.
(viii) $b=6$, $c=4$, $C=40°$, measure a.
(ix) $b=8$, $c=6$, $C=65°$, measure a.
(x) $A=40°$, $B=60°$, $C=80°$, measure a.
(xi) $A=50°$, $B=40°$, $C=70°$, measure a.
(xii) $A=125°$, $b=7{\cdot}3$, $c=5{\cdot}4$, measure a.
(xiii) $A=90°$, $a=11{\cdot}2$, $b=7{\cdot}3$, measure c.
(xiv) $a=b=6{\cdot}9$, $A=50°$, measure c.
(xv) $a=2b$, $c=\frac{3b}{2}$, measure A.

2. Draw two unequal lines, AC, BD bisecting each other; join AB, BC, CD, DA and measure them. ABCD is a *parallelogram*.

3. Draw two equal lines AC, BD bisecting each other; join AB, BC, CD, DA; measure $\angle$ABC. ABCD is a *rectangle*.

4. Draw two unequal lines AC, BD bisecting each other at right angles; join AB, BC, CD, DA and measure them. ABCD is a *rhombus*.

5. Draw two equal lines AC, BD bisecting each other at right angles; join AB, BC, CD, DA; measure AB, BC, $\angle$ABC. ABCD is a *square*.

6. Draw two unequal perpendicular lines AC, BD such that AC bisects BD; join AB, BC, CD, DA and measure them. ABCD is a *kite*.

7. Draw an angle of 57° and cut off AB, AC from the arms of the angle so that AB=5 cm., AC=8 cm.; construct a point D such that BD=AC and CD=AB. What sort of a quadrilateral is ABCD?

8. Construct a parallelogram ABCD, given AB=7 cm., AC=10 cm., BD=8 cm.; measure BC, CD.

9. Construct an isosceles triangle with a base of 6 cm. and a vertical angle of 70°; measure its sides.

10. Construct a rhombus ABCD, given AB=5 cm., AC=6 cm.; measure $\angle$BAD.

11. Construct an isosceles triangle of base 4·6 cm. and height 5 cm.; measure its vertical angle.

12. Construct the quadrilateral ABCD, given AB=BC=3 cm., AD=DC=5 cm., $\angle$ABC=120°; measure $\angle$ADC.

13. Construct the rhombus ABCD, given AC=6 cm., BD=9 cm. measure AB.

14. Construct the rhombus ABCD, given $\angle$ABC=40°, BD=7 cm., measure AC.

15. Construct a rectangle ABCD, given BD=8 cm. and that AC makes an angle of 54° with BD; measure AB, BC.

16. Construct a trapezium ABCD with AB, CD its parallel sides such that AB =8, BC=4, CD=3, AD=2; measure $\angle$BAD.

17. Construct the quadrilateral ABCD, given that

(i) AB=4, BC=4·5, CD=3, $\angle$ABC=80°, $\angle$BCD=110°; measure AD.

(ii) AB=5, AC=6, AD=4, BD=7, CD=3; measure BC.

(iii) $\angle$ABC=70°, $\angle$BCD=95°, $\angle$CDA=105°, AB=5, AD=4; measure BC.

(iv) AB=5, BC=6, CD=3, DA=4·5, $\angle$ADC=100°; measure $\angle$ABC.

(v) AB=5, $\angle$CAB=35°, $\angle$ABD=47°, $\angle$ACB=65°, $\angle$ADB=54°; measure CD.

18. Construct the triangle ABC, given that

(i) $a+b=11$, $b+c=16$, $c+a=13$; measure A.
(ii) $A-B=25°$, $C=55°$, $c=7$; measure a
(iii) $A : B : C=1 : 2 : 3$, $a=3$; measure c.
(iv) $A+B=118°$, $B+C=96°$, $a=7$; measure c.

19. Construct an equilateral triangle ABC such that if D is a point on BC given by BD=3 cm., then $\angle DAC=40°$; measure BC.

20. Construct a square having one diagonal 5 cm.; measure its side.

21. AD is an altitude of the triangle ABC; given AD=4 cm., $\angle ABC=55°$, $\angle ACB=65°$, construct $\triangle ABC$; measure BC.

22. AE is a median of the triangle ABC; given AB=4 cm., AC=7 cm., AE=4·5 cm., construct $\triangle ABC$; measure BC.

23. AD is an altitude of the triangle ABC; given AB=6 cm., AD=4 cm., $\angle ACB=68°$, construct $\triangle ABC$; measure BC.

24. AD is an altitude of $\triangle ABC$; AD=4 cm., $\angle BAC=75°$, $\angle ABC=50°$, construct $\triangle ABC$; measure BC.

25. The distance between the opposite sides of a parallelogram are 3 cm., 4 cm., and one angle is 70°; construct the parallelogram and measure one of the longer sides.

26. Construct a parallelogram of height 4 cm., having its diagonals 5 cm., 8 cm. in length: measure one of the longer sides.

27. Construct an equilateral triangle of height 4 cms.; measure its side.

28. Construct the triangle ABC, given that

(i) $a+b=2c=14$, $A=70°$; measure a.
(ii) $a+b+c=20$, $A=65°$, $B=70°$; measure a.
(iii) $a=10$, $b+c=13$, $A=80°$; measure b.
(iv) $a=8$, $b+c=10$, $B=35°$; measure b.
(v) $a=9$, $c-b=4$, $B=25°$; measure c.
(vi) $a=9$, $b-c=2$, $A=70°$; measure b.
(vii) $a=5$, $b=3$, $A-B=20°$; measure c.

29. Construct an isosceles triangle of height 5 cm. and perimeter 18 cm.; measure its base.

30. Each of the base angles of an isosceles triangle exceeds the vertical angle by 24°; the base is 4 cms.; construct the triangle and measure its other sides.

Equal Intercept Theorems.

THEOREM 20.

The straight line drawn through the middle point of one side of a triangle parallel to another side bisects the third side.

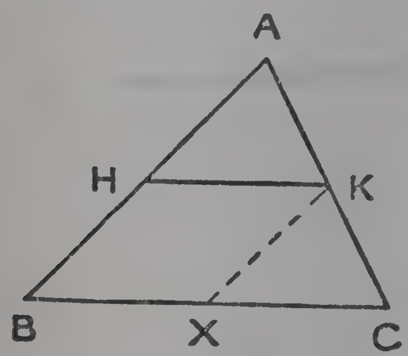


FIG. 83.

Given AH = HB and HK is parallel to BC.

To prove AK = KC.

Through K draw a line KX parallel to AB to cut BC at X.

Since HK is parallel to BX, given, and KX is parallel tc HB constr.

$\therefore$ BHKX is a parallelogram.

$\therefore$ KX = HB.

But AH = HB, given. $\therefore$ KX = AH.

$\therefore$ in the $\triangle$s AHK, KXC,

$\angle$HAK = $\angle$XKC, corresp. $\angle$s., AH$\|$KX.

$\angle$HKA = $\angle$XCK, corresp. $\angle$s., HK$\|$XC.

AH = KX.

$\therefore$ $\triangle$AHK $\equiv$ $\triangle$KXC (2 angles, corr. side).

$\therefore$ AK = KC.

Q.E.D.

THEOREM 21.

The straight line joining the middle points of two sides of a triangle is parallel to the base and equal to half the base.

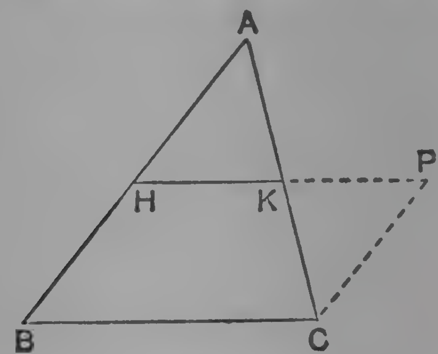


FIG. 84.

Given H, K are the middle points of AB, AC.

To prove HK is parallel to BC and $HK = \frac{1}{2}BC$.

Through C, draw CP parallel to BA to meet HK produced at P.

In the $\triangle$s AHK, CPK,

$\angle AHK = \angle CPK$, alt. $\angle$s., $HA \parallel CP$.

$\angle HAK = \angle PCK$, alt. $\angle$s., $HA \parallel CP$.

$AK = KC$, given.

$\therefore \triangle AHK \equiv \triangle CPK$ (2 angles, corr. side).

$\therefore CP = AH$.

But $AH = BH$, given.

$\therefore CP = BH$.

Also CP is drawn parallel to BH.

$\therefore$ the lines CP, BH are equal and parallel.

$\therefore$ BCPH is a parallelogram.

$\therefore$ HK is parallel to BC.

Also $HK = KP$ from congruent triangles.

$\therefore HK = \frac{1}{2}HP$.

But $HP = BC$ opp. sides of parallelogram.

$\therefore HK = \frac{1}{2}BC$.

Q.E.D

THEOREM 22.

If there are three or more parallel straight lines, and if the intercepts made by them on any straight line cutting them are equal, then the intercepts made by them on any other straight line that cuts them are equal.

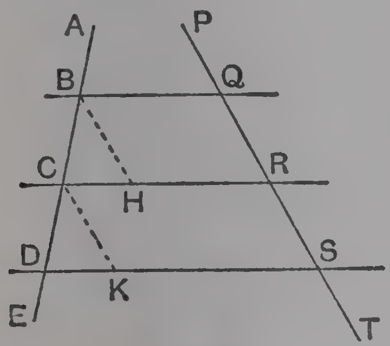


FIG. 85.

Given three parallel lines cutting a line AE at B, C, D and any other line PT at Q, R, S and that $BC = CD$.

To prove $QR = RS$.

Draw BH, CK parallel to PT to meet CR, DS at H, K.

Then BH is parallel to CK.

$\therefore$ in the $\triangle$s BCH, CDK,

$$\angle CBH = \angle DCK, \text{ corresp. } \angle\text{s., } BH \parallel CK.$$

$$\angle BCH = \angle CDK, \text{ corresp. } \angle\text{s., } CH \parallel DK.$$

$$BC = CD, \text{ given.}$$

$$\therefore \triangle BCH \equiv \triangle CDK \text{ (2 angles, corr. side).}$$

$$\therefore BH = CK.$$

But BQRH is a $\parallel$gram since its opposite sides are parallel.

$$\therefore BH = QR.$$

And CRSK is a $\parallel$gram since its opposite sides are parallel.

$$\therefore CK = RS.$$

$$\therefore QR = RS.$$

Q.E.D.

EXERCISE XXVI.

1. ABC is a $\triangle$; H, K are the mid-points of AB, AC ; P is any point on BC ; prove HK bisects AP.

2. In $\triangle ABC$, $\angle BAC=90°$; D is the mid-point of BC ; prove that $AD=\frac{1}{2}BC$. (From D, drop a perpendicular to AC.)

3. In Fig. 86, if $AC=CB$ and if AP, BQ, CR are parallel, prove that $CR=\frac{1}{2}(AP+BQ)$.

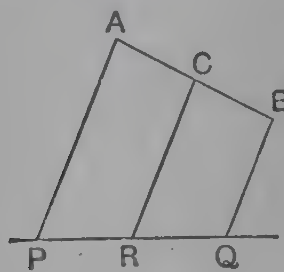


FIG. 86.

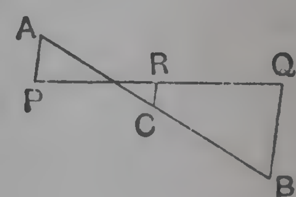


FIG. 87.

4. In Fig. 87, if $AC=CB$, and if AP, BQ, CR are parallel, prove that $CR=\frac{1}{2}(BQ-AP)$.

5. P, Q, R, S are the mid-points of the sides AB, BC, CD, DA of the quadrilateral ABCD ; prove that PQ is equal and parallel to SR.

6. In $\triangle ABC$, $\angle ABC=90°$; BCX is an equilateral triangle ; prove that the line from X parallel to AB bisects AC.

7. ABC is a $\triangle$; H, K are the mid-points of AB, AC ; BK, CH are produced to X, Y so that $BK=KX$ and $CH=HY$; prove that $XY=2BC$.

8. O is a fixed point ; P is a variable point on a fixed line AB ; find the locus of the mid-point of OP.

9. O is a fixed point ; P is a variable point on a fixed circle, centre A ; prove that the locus of the mid-point of OP is a circle whose centre is at the mid-point of OA.

10. Prove that the lines joining the mid-points of opposite sides of any quadrilateral bisect each other.

11. If the diagonals of a quadrilateral are equal and cut at right angles, prove that the mid-points of the four sides are the corners of a square.

12. ABC is a $\triangle$; AX, AY are the perpendiculars from A to the bisectors of the angles ABC, ACB ; prove that XY is parallel to BC.

13. AD, BE are altitudes of $\triangle$ABC and intersect at H; P, Q, R are the mid-points of HA, AB, BC; prove that $\angle PQR = 90°$.

14. ABCD is a quadrilateral, having AB parallel to CD; P, Q, R, S are the mid-points of AD, BD, AC, BC; prove that (i) PQ=RS; (ii) $PS = \frac{1}{2}(AB + CD)$; (iii) $QR = \frac{1}{2}(AB \sim CD)$.

15. ABC is a $\triangle$; D is the mid-point of BC; P is the foot of the perpendicular from B to the bisector of $\angle$BAC; prove that $DP = \frac{1}{2}(AB \sim AC)$.

16. ABC is a $\triangle$; D is the mid-point of BC; Q is the foot of the perpendicular from B to the external bisector of $\angle$BAC; prove that $DQ = \frac{1}{2}(AB + AC)$.

17. ABCD is a quadrilateral having AB=CD; P, Q, R, S are the mid-points of AD, AC, BD, BC; prove that PS is perpendicular to QR.

18. In Fig. 88, if BD=DC and AP=AQ, prove that BP=CQ and $AP = \frac{1}{2}(AB + AC)$.

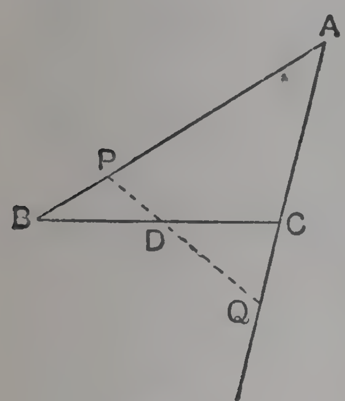


FIG. 88.

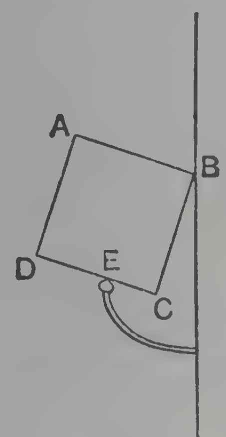


FIG. 89.

19. A square box ABCD, each edge 13″, rests in the rack of a railway carriage and against the wall: the point of contact E is 11″ from the wall; CE=ED. Prove that C is 5″ from the wall, and find the distances of A, D from the wall.

20. ABC is a $\triangle$; E, F are the mid-points of AC, AB; BE cuts CF at G; AG is produced to X so that AG=GX and cuts BC at D; prove that (i) GBXC is a parallelogram; (ii) $DG = \frac{1}{2}GA = \frac{1}{3}DA$.

21. ABCD is a parallelogram; XY is any line outside it; AP, BQ, CR, DS are perpendiculars from A, B, C, D to XY; prove that AP+CR=BQ+DS.

22. The diagonals AC, BD of the square ABCD intersect at O; the bisector of $\angle$BAC cuts BO at X, BC at Y; prove that CY=2OX.

CONSTRUCTION 11.

Divide a given straight line into any given number of equal parts.

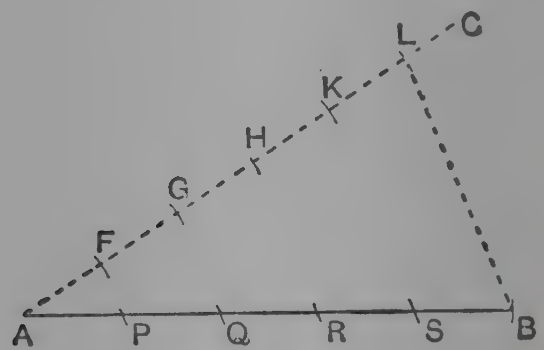


FIG. 90.

Given a line AB.

To construct points dividing AB into any number (say 5) equal parts.

Through A, draw any line AC.

Along AC, step out with compasses equal lengths, the number of such lengths being the required number of equal parts (in this case 5).

Let the equal lengths be AF, FG, GH, HK, KL.

Join LB, and through F, G, H, K draw lines parallel to BL, meeting AB at P, Q, R, S.

Then AP, PQ, QR, RS, SB are the required equal parts.

Proof. Since the parallel lines FP, GQ, HR, KS, LB cut off equal intercepts on AC, they cut off equal intercepts on AB.

Q.E.F.

EXERCISE XXVII.

1. Draw a line AB; divide it into three equal parts without measuring it.

2. Draw a line AB and construct a point P on AB such that $\frac{AP}{PB}=\frac{2}{3}$.

3. Draw a line AB and construct a point Q on AB produced, such that $\frac{AQ}{BQ}=\frac{7}{4}$.

4. Divide a given line in the ratio 5 : 3 both internally and externally.

5. Construct a diagonal scale which can be used for measuring lengths to $\frac{1}{100}$ inch.

6. By using a diagonal scale, draw a line of length 2·73 inches: on this line as base construct an isosceles right-angled triangle and measure its equal sides as accurately as possible.

7. Use a diagonal scale to measure the hypotenuse of a right-angled triangle whose sides are 2″ and 3″.

8. On a scale of 6″ to the mile, what length represents 2000 yards? Draw a scale showing hundreds of yards.

9. What is the R.F. (*i.e. representative fraction*) for a map of scale 2″ to the mile? Construct a scale for reading off distances up to 5000 yards, and as small as 500 yards.

10. The R.F. of a map is 1 : 20,000; express this in inches to the mile, and construct a suitable scale to read miles and tenths of miles.

11. Given two lines AB, AC and a point P between them, construct a line through P, cutting AB, AC at Q, R so that QP=PR.

12. Given two lines AB, AC and a point P between them, construct a line through P with its extremities on AB, AC and divided at P in the ratio 2 : 3.

MISCELLANEOUS CONSTRUCTIONS.

EXERCISE XXVIII.

1. Draw an angle BAC and a line PQ; construct points R, S on AB, AC such that RS is equal and parallel to PQ.

2. Draw a circle and construct points P, Q, R on it such that PQ=QR=RP; take any other point X on the circle. Measure XP, XQ, XR, and verify that the longest of these equals the sum of the other two.

3. Draw an angle BAC of 50°; construct on AB, AC points P, Q such that ∠QPA=90° and PQ=4 cm. Measure AP.

4. Draw a large quadrilateral ABCD, so that AB is not parallel to CD; construct a point P such that PA=PB and PC=PD.

5. Draw a line AB and take a point C distant 2″ from AB; construct a circle with C as centre, cutting AB at two points 3″ apart. Measure its radius.

6. Draw an angle BAC of 70°; construct a point P whose distances from AB, AC are 3 cm., 4 cm. Measure AP.

7. Draw a line AB and take a point C distant 2″ from AB; construct two points P, Q, each of which is $1\frac{1}{2}$″ from AB and $1\frac{1}{2}$″ from C. Measure PQ.

8. Draw two lines AB, AC and take a point P somewhere between them; construct a line to pass through P and cut off equal lengths from AB and AC.

9. Draw two lines AB, CD and take any point E between them. Construct a line to pass through E and the (inaccessible) point of intersection of AB, CD. (Use the system of parallel lines shown in Fig. 91.)

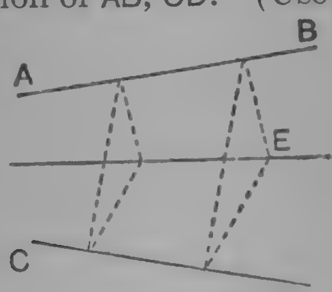


FIG. 91.

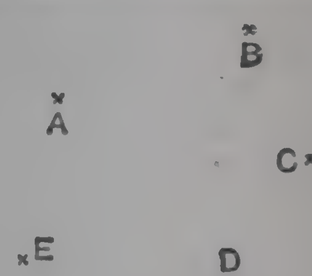


FIG. 92.

10. Draw a circle and take two points A, B outside it. Construct a circle to pass through A, B and have its centre on the first circle. When is this impossible?

11. Construct five points in the same relative position to each other as are A, B, C, D, E in Fig. 92.

12. Take a line AB and a point C outside it such that the foot of the perpendicular from C to AB would be off the page. Construct that portion of the perpendicular which comes on the page.

13. Take a line AB and a point C and suppose there is an obstacle between C and AB which a set square cannot move over (see Fig. 93). Construct a line through C parallel to AB.

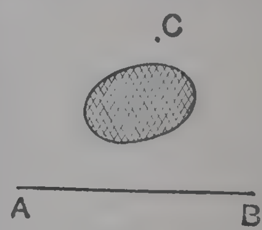


FIG. 93.

14. By folding, obtain a crease which (i) bisects a given angle, (ii) bisects a given line at right angles.

15. By folding, obtain the perpendicular to a given line from a given point outside it.

16. By folding, obtain an angle of 45°.

17. Take a triangular sheet of paper and find by folding the point which is equidistant from the three corners.

18. Given two points H, K on the same side of a given line AB, construct a point P on AB such that PH, PK make equal angles with AB.

19. Given two points H, K on opposite sides of a given line CD, (see Fig. 94), construct a point P on CD such that $\angle HPC = \angle KPC$.

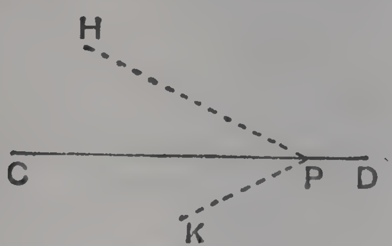


FIG. 94.

20. Given a triangle ABC, construct a line parallel to BC, cutting AB, AC at H, K such that BH+CK=HK.

21. Given a triangle ABC, construct a rhombus with two sides along AB, AC and one vertex on BC.

22. Given two parallel lines AB, CD and a point P between them, construct a line through P, cutting AB, CD at Q, R such that QR is of given length.

23. Given a triangle ABC, construct a point which is equidistant from B and C and also equidistant from the lines AB and AC.

24. By construction and measurement, find the height of a regular tetrahedron, each edge of which is 2″.

25. A room is 20 feet long, 15 feet wide, 10 feet high ; a cord is stretched from one corner of the floor to the opposite corner of the ceiling. find by drawing and measurement the angle which the cord makes with the floor.

26. Copy the following Figs. 95-109 on any convenient scale:

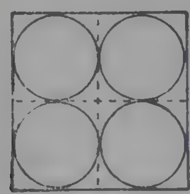


FIG. 95.

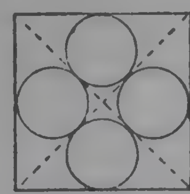


FIG. 96.

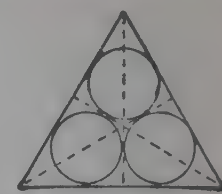


FIG. 97.

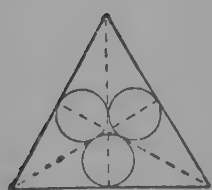


FIG. 98.

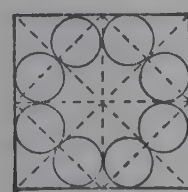


FIG. 99.

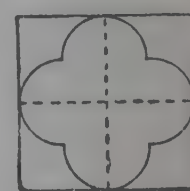


FIG. 100.

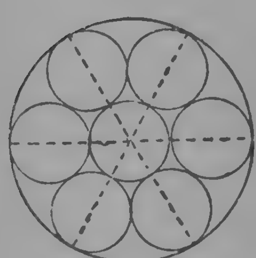


FIG. 101.

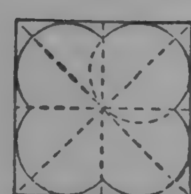


FIG. 102.

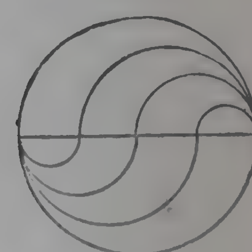


FIG. 103.

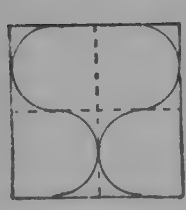


FIG. 104.

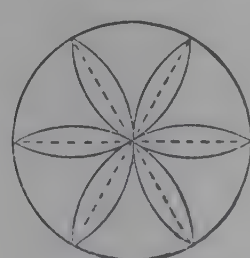


FIG. 105.

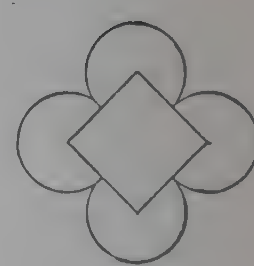


FIG. 106.

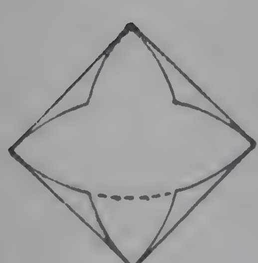


FIG. 107.

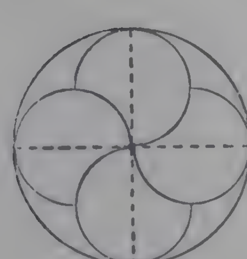


FIG. 108.

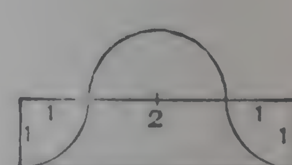


FIG. 109.

REVISION PAPERS.

SECTION I.

1.

1. What angle is equal to (i) $\frac{1}{5}$ of its supplement ?
(ii) $\frac{1}{5}$ of its complement ?

2. ABCD is a straight line such that $AB=\frac{1}{3}AC=\frac{1}{5}AD$; $CD=2''$; calculate the distance between the mid-points of AB and AD.

3. ABC is a triangle ; AB is produced to D, BC is produced to E ; $\angle BAC=74°$, $\angle ABC=52°$; calculate the angle between the bisectors of the angles DBC, ECA.

4. If in $\triangle ABC$, $AB=AC$ and $\angle BAC=20°$, and if D is a point on AC such that $\angle DBC=60°$, prove $AD=DB$.

2.

1. (i) What is the acute angle between E. by N. and N. by E. ?
(ii) What is the reflex angle between N.W. and W. ?

2. A line AB, 3″ long is produced to points P, Q such that $AP=4PB$, and $AQ=3QB$. What is the length of PQ ?

3. Calculate the angle x in Fig. 110.

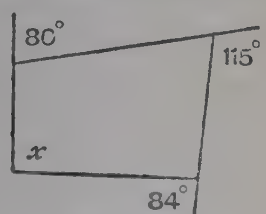


FIG. 110.

4. ABC is a triangle ; D is a point on BC such that $\angle CAD=\angle ABC$; prove $\angle ADC=\angle BAC$.

3.

1. (i) If in Fig. 17, p. 19, $\angle ACE=4\angle BCE$, calculate $\angle BCE$.
(ii) If in Fig. 18, p. 19, a exceeds x by 90°, calculate a.

2. Fig. 111 represents two concentric circles ; the centre lies on AB produced and on CD produced ; prove AB=CD.

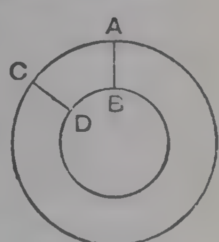


FIG. 111.

3. ABCD is a square ; AXB is an equilateral triangle outside the square ; prove that

$$\angle ACX=\tfrac{1}{2}\angle ABX.$$

4. The triangle ABC is right-angled at B ; X is a point on AC such that XB=XC ; prove that XB=XA.

4.

1. A is due E. of B; P is N. 17° W. of A and N. 29° E. of B; calculate $\angle APB$.

2. Fig. 112 represents a "Pentagram" (*i.e.* the inner figure is a regular pentagon); calculate the acute angle at each corner.

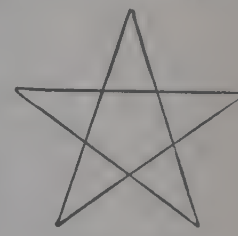


FIG. 112.

3. X is a point inside the triangle ABC such that $\angle XAB = \angle XCA$; prove

$$\angle AXC + \angle BAC = 180°.$$

4. A, B, C, D are four points equidistant from a fifth point O; prove $\angle ABC + \angle ADC = \angle BAD + \angle BCD$.

5.

1. If all the marked angles in Fig. 113 are equal, calculate the size of each.

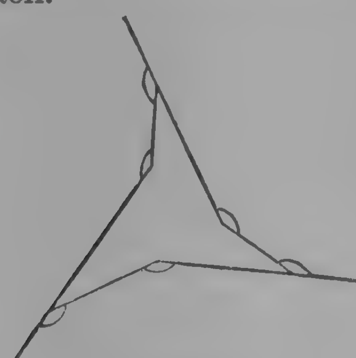


FIG. 113.

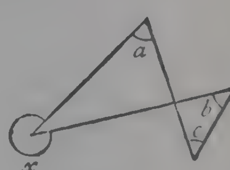


FIG. 114.

2. With the data in Fig. 114, express x in terms of a, b, c.

3. AB, DC are the parallel sides of the trapezium ABCD; if AD = DC, prove that AC bisects $\angle BAD$.

4. In $\triangle ABC$, AB = AC and $\angle BAC = 90°$; D is any point on CB; BH, CK are the perpendiculars from B, C to AD (produced if necessary); prove that

(i) BH = AK, (ii) BH ~ CK = HK.

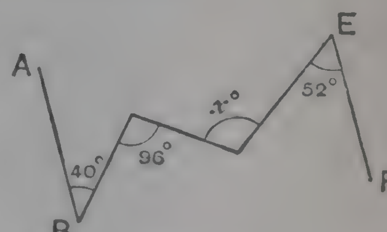


FIG. 115.

6.

1. If, in Fig. 115, AB is parallel to EF, calculate the angle x.

2. In $\triangle ABC$, AB = AC; BC is produced to D so that CD = AB; prove $\angle ABD = 2\angle ADB$.

3. What is the name of a quadrilateral in which:
 (i) The diagonals bisect each other?
 (ii) The diagonals bisect each other at right angles?
 (iii) The diagonals are equal and bisect each other?
 (iv) One pair of opposite sides are equal and the other pair are parallel and unequal?

4. ABC is a triangle; APQB, AXYC are squares outside $\triangle ABC$; prove that (i) $\triangle PAC \equiv \triangle BAX$; (ii) PC is perpendicular to BX.

7.

1. It requires four complete turns of the handle to wind up a bucket from the bottom of a well 24 feet deep. Through what angle must the handle be turned to raise the bucket 5 feet.

2. The angles of a triangle are in the ratio 1 : 3 : 5. Find them.

3. ACB is a straight line; ABX, ACY are equilateral triangles on opposite sides of AB; prove $CX=BY$.

4. ABCD is a quadrilateral; ADCX, BCDY are parallelograms; prove that XY bisects AB.

8.

1. If the reflex angle AOB is four times the acute angle AOB, find $\angle AOB$.

2. In $\triangle ABC$, $\angle BAC=44°$, $\angle ABC=112°$; find the angle between the lines which bisect $\angle ABC$ and $\angle ACB$.

3. The base BC of an isosceles triangle ABC is produced to D so that $CD=CA$, prove $\angle ABD=2\angle ADB$.

4. ABCD is a parallelogram; P is the mid-point of AB; CP and DA are produced to meet at Q; DP and CB are produced to meet at R; prove $QR=CD$.

9.

1. $\angle AOB=x°$; AO is produced to C; OP bisects $\angle BOC$; OQ bisects $\angle AOB$; calculate reflex angle POQ.

2. In $\triangle ABC$, $\angle ABC=35°$, $\angle ACB=75°$; the perpendiculars from B, C to AC, AB cut at Q. Find $\angle BOC$.

3. The bisector of the angle BAC cuts BC at D; through C a line is drawn parallel to DA to meet BA produced at P; prove $AP=AC$.

4. ABC is an acute-angled triangle; BAHK, CAXY are squares outside the triangle; prove that the acute angle between BH and CX equals $90°-\angle BAC$.

10.

1. Find the sum of the interior angles of a 15-sided convex polygon.

2. The sum of one pair of angles of a triangle is 100°, and the difference of another pair is 60°; prove that the triangle is isosceles.

3. ABC is a triangle right-angled at C; P is a point on AB such that $\angle PCB = \angle PBC$; prove $\angle PCA = \frac{1}{2}\angle BPC$.

4. O is a point inside an equilateral triangle ABC; OAP is an equilateral triangle such that O and P are on opposite sides of AB; prove $BP = OC$.

11.

1. If a ship travels due East or West one sea mile, her longitude alters 1 minute if on the equator, and 2 minutes if in latitude 60°. Find her longitude if she starts (i) at lat. 0°, long. 2° E. and steams 200 miles West; (ii) at lat. 60° N., long. 2° W. and steams 150 miles East.

2. The bisectors of $\angle$s ABC, ACB of $\triangle$ABC meet at O; if $\angle BOC = 135°$, prove $\angle BAC = 90°$.

3. In $\triangle$ABC, $\angle ACB = 3\angle ABC$; from AB a part AD is cut off equal to AC; prove $CD = DB$.

4. In $\triangle$ABC, $AB = AC$; from any point P on AB a line is drawn perpendicular to BC and meets CA produced in Q; prove $AP = AQ$.

12.

1. O is a point outside a line ABCD such that $OA = AB$, $OB = BC$, $OC = CD$; $\angle BOC = x°$; calculate $\angle OAD$ and $\angle ODA$ in terms of x.

2. In Fig. 116, if OQ bisects $\angle AOC$, prove $\angle BOC - \angle BOA = 2\angle QOB$.

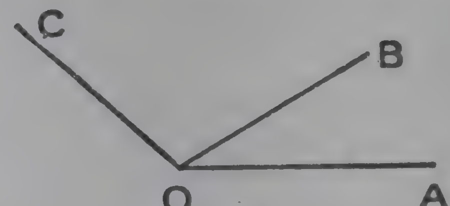


Fig. 116.

3. ABCD is a quadrilateral; $DA = DB = DC$; prove $\angle BAC + \angle BCA = \frac{1}{2}\angle ADC$.

4. ABCD is a parallelogram; BP, DQ are two parallel lines cutting AC at P, Q; prove BQ is parallel to DP.

13.

1. In $\triangle ABC$, $\angle BAC = 115°$, $\angle BCA = 20°$; AD is the perpendicular from A to BC; prove $AD = DB$.

2. In Fig. 117, AB is parallel to ED; prove that

$$\text{reflex } \angle EDC - \text{reflex } \angle ABC = \angle BCD.$$

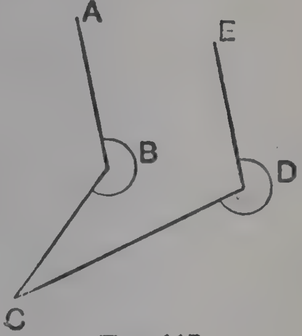


FIG. 117.

3. ABCD is a quadrilateral;

$$\angle ABC = \angle ADC = 90°;$$

prove that the bisectors of $\angle$s DAB, DCB are parallel.

4. In $\triangle ABC$, $\angle ABC = 90°$, $\angle ACB = 60°$; prove $AC = 2BC$.

14.

1. Two equilateral triangles ABC, AYZ lie outside each other; if $\angle CAY = 15°$, find the angle at which YZ cuts BC.

2. In $\triangle ABC$, $AB = AC$; D is a point on AC such that $DB = BC$; prove $\angle DBC = \angle BAC$.

3. The altitudes BD, CE of $\triangle ABC$ meet at H; if $HB = HC$, prove $AB = AC$.

4. P, Q, R, S are points on the sides AB, BC, CD, DA of a square; if PR is perpendicular to QS, prove $PR = QS$.

15.

1. In Fig. 118, express x in terms of a, b, c.

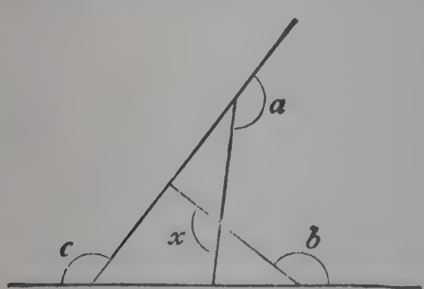


FIG. 118.

2. D is any point on the bisector of $\angle BAC$; DP, DQ are drawn parallel to AB, AC, to meet AC, AB at P, Q; prove $DP = DQ$.

3. ABC is a $\triangle$; D, E are points on BC such that $\angle BAD = \angle CAE$; if $AD = AE$, prove $AB = AC$.

4. ABCD is a square; the bisector of $\angle BCA$ cuts AB at P; PQ is the perpendicular from P to AC; prove $AQ = PB$.

16.

1. ABCDEFGH is a regular octagon; calculate the angle at which AD cuts BF.

2. In $\triangle ABC$, AD is perpendicular to BC and AP bisects $\angle BAC$; if $\angle ABC > \angle ACB$, prove $\angle ABC - \angle ACB = 2\angle PAD$.

3. ABCD is a straight line such that $AB = BC = CD$; BPQC is a parallelogram; if $BP = 2BC$, prove PD is perpendicular to AQ.

4. The sides AB, AC of $\triangle ABC$ are produced to D, E; AH, AK are lines parallel to the bisectors of $\angle$s BCE, CBD meeting BC in H, K: prove $AB + AC = BC + HK$.

17.

1. In Fig. 119, express z in terms of a, b, x, y.

2. AB, BC, CD, DE are successive sides of a regular n-sided polygon; find the angle between AB and DE.

3. In $\triangle ABC$, $AB = AC$; BA is produced to E; the bisector of $\angle ACB$ meets AB at D; prove $\angle CDE = \frac{3}{4}\angle CAE$.

4. In $\triangle ABC$, $\angle BAC = 90°$; O is the centre of the square BPQC external to the triangle; prove that AO bisects $\angle BAC$.

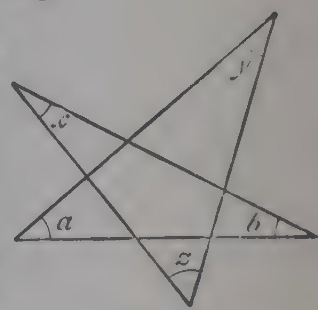


FIG. 119.

18.

1. B is 4 miles due East of A; a ship sailing from A to B against the wind takes the zigzag course shown in Fig. 120, her directions being alternately N. 30° E. and S. 30° E.; what is the total distance she travels?

FIG. 120.

2. ABC is a triangular sheet of paper, $\angle ABC = 40°$, $\angle ACB = 75°$; the sheet is folded so that B coincides with C; find the angle which the two parts of AB make with each other in the folded position.

3. In $\triangle ABC$, $AB = AC$; the bisector of $\angle ABC$ meets AC at D; P is a point on AC produced so that $\angle ABP = \angle ADB$; prove $BC = CP$.

4. ABC is a $\triangle$; BDEC is a square outside $\triangle ABC$; lines through B, C parallel to AD, AE meet at P; prove PA is perpendicular to BC.

PART II.

SECTION II.

AREAS.

Suppose a rectangle is x units long and y units wide, it can be divided up by lines parallel to the sides into compartments, each of which is a square whose side is of unit length.

The area of each of these squares is called a **unit of area.**

If in Fig. 121 there are y rows and each row contains x squares, then the whole rectangle contains xy units of area.

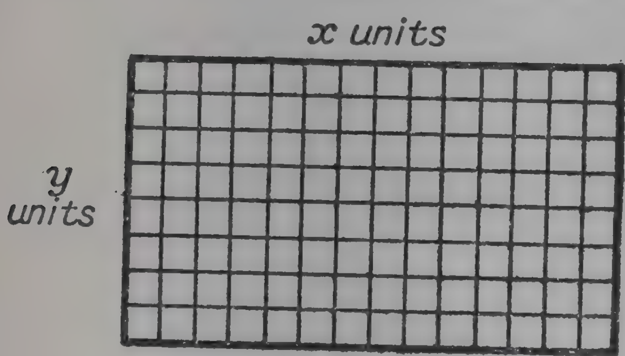


FIG. 121.

Consider a rectangle $2\frac{1}{2}$ inches long $1\frac{1}{3}$ inches wide. Take the unit of length as $\frac{1}{6}$ inch: then the length of the rectangle $=\frac{5}{2}$ inches $=\frac{15}{6}$ inches $=15$ units, and the breadth of the rectangle $=\frac{4}{3}$ inches $=\frac{8}{6}$ inches $=8$ units.

$\therefore$ the area of the rectangle $=15\times 8$ units of area.

But a unit of area is a square of side $\frac{1}{6}$ inch. Now a square of side 1 inch can be divided into 6×6 compartments each of which is a square of side $\frac{1}{6}$ inch.

$\therefore$ 6×6 units of area (square of side $\frac{1}{6}$ inch) $=1$ sq. inch.

∴ the area of the rectangle

$$=\frac{15\times 8}{6\times 6}\text{ sq. inches}=\frac{15}{6}\times\frac{8}{6}=2\tfrac{1}{2}\times 1\tfrac{1}{3}\text{ sq. inches.}$$

Hence we see that whether the sides of a rectangle are integers or fractions, the area is measured by the product of the measure of its sides. This proof fails for incommensurable lengths. This result is stated as follows.

THEOREM 23.

The area of a rectangle is measured by the product of the measures of two adjacent sides.

A right-angled triangle can be regarded as half a rectangle, because a rectangle is bisected by a diagonal.

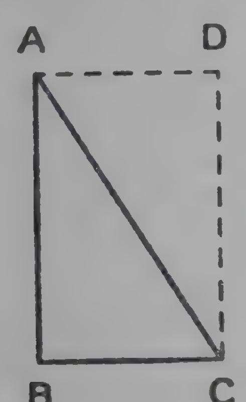


FIG. 122.

In Fig. 122 the right-angled triangle ABC is half the rectangle ABCD, and its area therefore equals $\frac{1}{2}$AB × BC.

EXERCISE XXIX.

1. Draw on squared paper a rectangle length 2·3 in., breadth 1·4 in., calculate its area and verify the result by counting.

2. Draw to scale a figure to illustrate the fact that (i) 1 sq. ft. =144 sq. in., (ii) 1 sq. cm.=100 sq. mm.

3. Draw an accurate figure to illustrate the fact that the area of a rectangle $2\frac{3}{4}$ in. long, $1\frac{1}{2}$ in. wide is $2\frac{3}{4}\times 1\frac{1}{2}$ sq. in.

4. A map is drawn on a scale of 4 miles to the inch.
   (i) What area is represented by $\frac{1}{2}$ sq. inch on the map ?
   (ii) What area on the map represents 1 sq. mile ?

5. Find the area of Figs. 123, 124, 125 (dimensions in inches) :

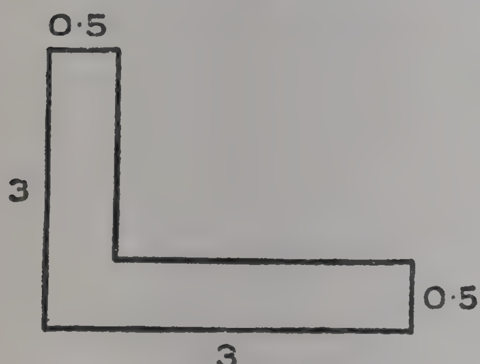


Fig. 123.

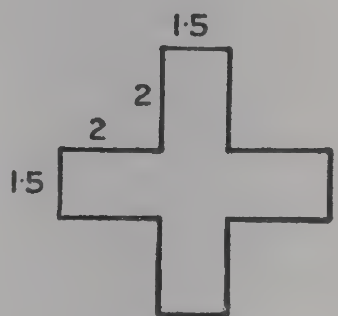


Fig. 124.

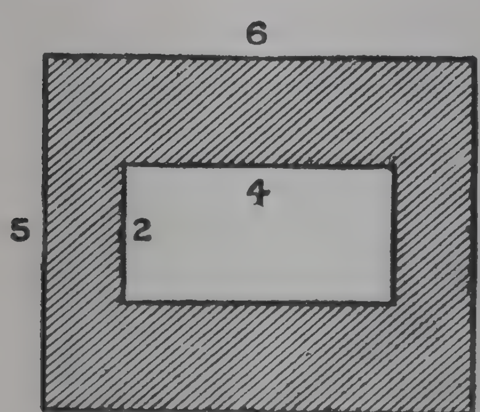


Fig. 125.

6. A rectangle is 10 inches long, and equals in area a square of side 5 inches ; what is its breadth ?

7. What is the length of the side of a square whose area equals that of a rectangle 2 feet long, 3 inches wide ?

8. Find the areas of Figs. 126, 127, 128, assuming the ruling shown is in fifths of an inch.

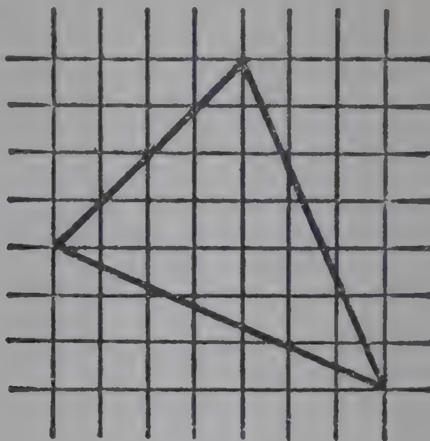


FIG. 126.

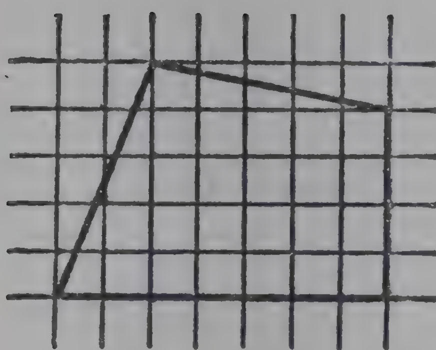


FIG. 127.

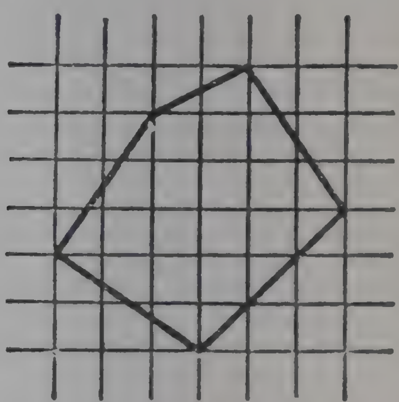


FIG. 128.

9. Find approximately (by counting small squares) the areas of Figs. 129, 130, assuming the ruling shown is in fifths of an inch.

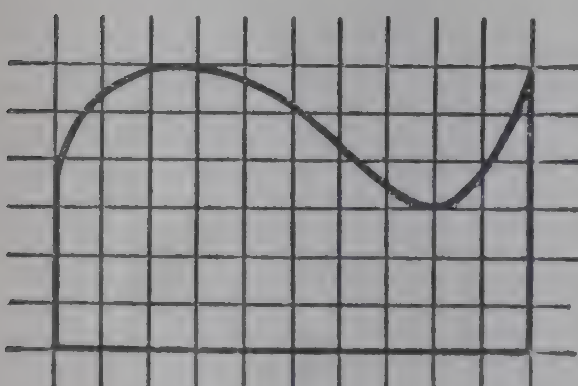


Fig. 129.

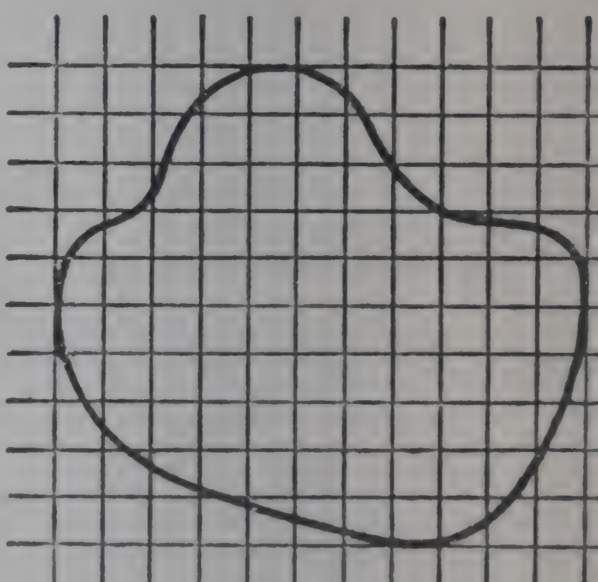


Fig. 130.

10. Draw on squared paper a circle of radius 2 inches, and find approximately its area by counting small squares.

11. What is the area of a right-angled triangle if the sides containing the right angle are 3 in., 4 in.? How do you prove this?

12. A is 100 yards East and 30 yards North of O; B is 40 yards East and 90 yards north of O. What is the area of the triangle OAB in sq. yards?

Definitions.

(i) If any side of a triangle is taken as its base, the perpendicular to that side from the opposite corner is called the **altitude** or **height**.

(ii) If any side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it is called the **altitude** or **height**.

A triangle has therefore three distinct altitudes AD, BE, CF (Fig. 131).

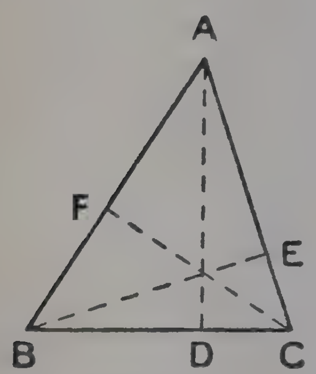


FIG. 131.

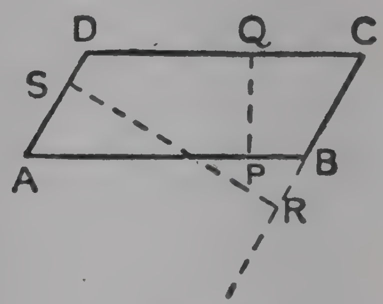


FIG. 132.

A parallelogram has two distinct altitudes PQ, RS (Fig. 132).

(iii) If two figures are of equal area, they are said to be **equivalent**.

THEOREM 24.

The area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels.

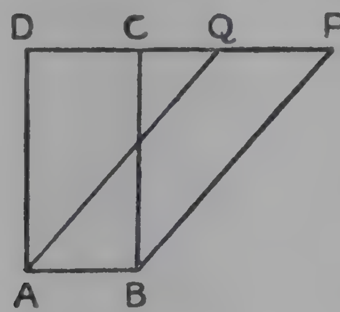


FIG. 133.

Given ABCD is a rectangle and ABPQ is a parallelogram on the same base AB and between the same parallels AB, DP.

To prove that the area ABCD = the area ABPQ.

In the $\triangle$s AQD, BPC,

$$\angle AQD = \angle BPC, \text{ corresp. } \angle\text{s., } AQ \parallel BP.$$
$$\angle ADQ = \angle BCP, \text{ corresp. } \angle\text{s., } AD \parallel BC.$$
$$AD = BC, \text{ opp. sides } \parallel\text{gram.}$$
$$\therefore \triangle AQD \equiv \triangle BPC \text{ (2 angles, corr. side).}$$

From the whole figure ABPD, subtract in succession each of the equal triangles AQD, BPC.

$\therefore$ the remaining figures ABPQ, ABCD are equal in area.

Q.E.D.

Corollary 1. The area of a parallelogram is measured by the product of its base and its altitude.

$$\begin{aligned}\text{The area of } ABPQ &= \text{the area of } ABCD = AB \times BC\\ &= \text{base } AB \times \text{height } BC.\end{aligned}$$

Corollary 2. Parallelograms on the same or equal bases and of equal altitudes are equal in area.

This follows from Corollary 1.

Note.—The proof of Theorem 24 applies, word for word, to any two parallelograms on the same base and between the same parallels.

THEOREM 25.

The area of a triangle is equal to one-half of the area of a rectangle on the same base and between the same parallels.

Given a triangle ABC and a rectangle PQBC on the same base BC, and between the same parallels BC, QA.

To prove that the area of ABC = half the area of PQBC.

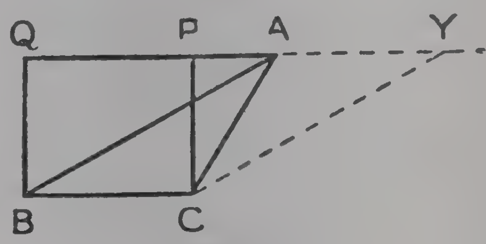


FIG. 124.

Complete the parallelogram ABCY.

Then area ABC = $\frac{1}{2}$ area ABCY, diagonal bisects ||gram, but area ABCY = area QBCP, same base BC, same parallels BC, QY.

$\therefore$ area ABC = $\frac{1}{2}$ area QBCP. Q.E.D.

Corollary 1. *The area of a triangle is measured by one-half the product of the measures of its base and its altitude.*

The area of QBCP = BC × CP.

But the perpendicular from A to BC equals CP, opp. sides of a rectangle.

$\therefore$ area ABC = $\frac{1}{2}$BC × CP = $\frac{1}{2}$ base × altitude.

Corollary 2. *Triangles on the same or equal bases and of equal altitudes are equal in area.*

This follows from Corollary 1.

Corollary 3. *If triangles of the same area have the same or equal bases, their altitudes are equal.*

This follows from Corollary 1.

Corollary 4. *If two triangles of the same area stand on the same side of the same base or of equal bases in the same straight line they are between the same parallels.*

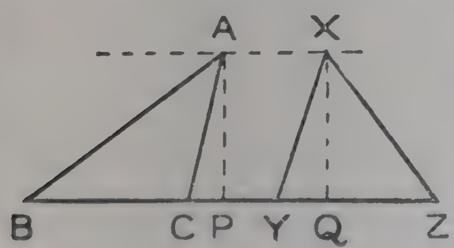


FIG. 135.

By Corollary 3, the altitudes AP, XQ are equal.

Also AP is parallel to XQ since each is perpendicular to BQ.

$\therefore$ AP is equal and parallel to XQ.

$\therefore$ AX is parallel to PQ or BZ.

THEOREM 26.

If a triangle and a parallelogram are on the same base and between the same parallels, the area of the triangle is equal to half that of the parallelogram.

Given the triangle ABC and the parallelogram ABXY on the same base AB and between the same parallels AB, CX.

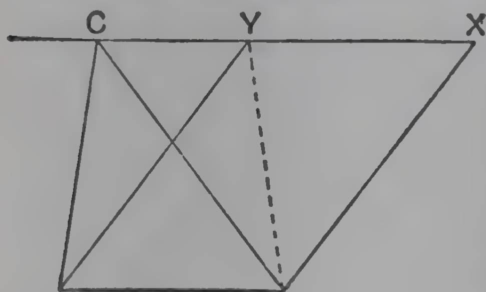


FIG. 136.

To prove $\triangle$ABC $= \frac{1}{2}$ ||gram ABXY.

Join BY.

The $\triangle$s ABC, ABY are on the same base and between the same parallels.

$\therefore$ $\triangle$ABC $=$ $\triangle$ABY in area.

Since the diagonal BY bisects the ||gram ABXY,

$\triangle$ABY $= \frac{1}{2}$ ||gram ABXY ;

$\therefore$ $\triangle$ABC $= \frac{1}{2}$ ||gram ABXY. Q.E.D.

The following formula for the area of a triangle is important :

If a, b, c are the lengths of the sides of a triangle and if $s = \frac{1}{2}(a+b+c)$, the area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

The Area of a Trapezium.

ABCD is a trapezium with AB parallel to DC ; DE is the perpendicular from D to AB.

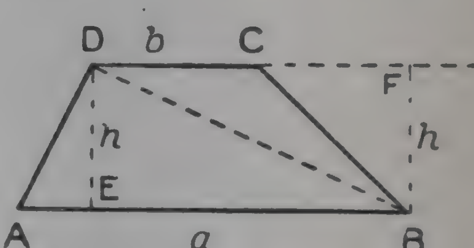


FIG. 137.

If AB $= a$ in., DC $= b$ in., DE $= h$ in., the area of the trapezium ABCD $= \frac{1}{2}h(a+b)$ sq. in.

Join DB, draw BF perpendicular to DC, produced.

Then BF $=$ DE $= h$ in., opp. sides of a ||gram.

Area of $\triangle$ADB $= \frac{1}{2}$DE . AB $= \frac{1}{2}ha$ sq. in.

Area of $\triangle$DCB $= \frac{1}{2}$BF . DC $= \frac{1}{2}hb$ sq. in. ;

$\therefore$ area of ABCD $= \frac{1}{2}ha + \frac{1}{2}hb = \frac{1}{2}h(a+b)$ sq. in. Q.E.D.

In words, the area of a trapezium is measured by the product of half the sum of the parallel sides and the distance between them.

EXERCISE XXX.

In Fig. 138, AD, BE, CF are altitudes of the triangle ABC.

1. In $\triangle ABC$, $\angle ABC=90°$, $AB=3''$, $BC=5''$; find area of ABC.

2. In Fig. 138, $AD=7''$, $BC=5''$; find area of ABC.

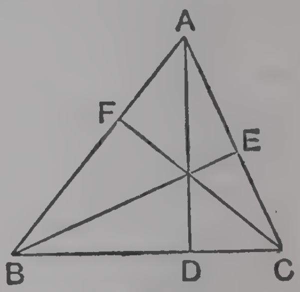


FIG. 138.

3. In Fig. 138, $BE=5''$, $AB=6''$, $CF=4''$; find AC.

4. In Fig. 138, $AD=6x''$, $BE=4x''$, $CF=3x''$, and the perimeter of ABC is 18″. Find BC.

5. In quad. ABCD, $AB=12''$, $BC=1''$, $CD=9''$, $DA=8''$, $\angle ABC=\angle ADC=90°$; find the area of ABCD.

6. In quad. ABCD, $AC=8''$, $BD=11''$, and AC is perpendicular to BD; find the area of ABCD.

7. Find the area of a triangle whose sides are 3″, 4″, 5″.

In Fig. 139, ABCD is a parallelogram; AP, AQ are the perpendiculars to BC, CD.

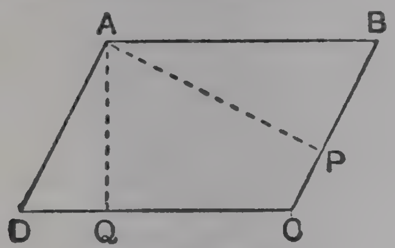


FIG. 139.

8. In Fig. 139, $AB=7''$, $AQ=3''$; find the area of ABCD.

9. In Fig. 139, $AB=5''$, $AP=4''$, $AD=6''$; find AQ.

10. In Fig. 139, $AP=3''$, $AQ=2''$, and perimeter of ABCD is 20″; find its area.

11. In quad. ABCD, $BC=8''$, $AD=3''$, and BC is parallel to AD; if the area of $\triangle ABC$ is 18 sq. in., find the area of $\triangle ABD$.

12. In quad. ABCD, $AB=5''$, $BC=3''$, $CD=2''$, $\angle ABC=\angle BCD=90°$; find the area of ABCD.

13. In Fig. 138, $AB=8''$, $AC=6''$, $BE=5''$; find CF.

14. The area of $\triangle ABC$ is 36 sq. cms., $AB=8$ cms., $AC=9$ cms., D is the mid-point of BC; find the lengths of the perpendiculars from D to AB, AC.

15. In the parallelogram ABCD, $AB=8''$, $BC=5''$; the perpendicular from A to CD is 3″; find the perpendicular from B to AD.

16. Find the area of a rhombus whose diagonals are 5″, 6″.

17. In △ABC, ∠ABC=90°, AB=6″, BC=8″, CA=10″; D is the mid-point of AC. Calculate the lengths of the perpendiculars from B to AC and from A to BD.

18. On an Ordnance Map, scale 6 inches to the mile, a football field is approximately a square measuring $\frac{1}{2}$ inch each way. Find the area of the field in acres, correct to $\frac{1}{10}$ acre.

19. Fig. 140 represents on a scale of 1″ to the foot a trough and the depth of the water in it. The water is running at 4 miles an

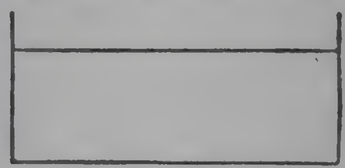


FIG. 140.

hour; find the number of gallons which pass any point in a minute to nearest gallon, taking 1 cub. ft.=$6\frac{1}{4}$ gallons.

20. Fig. 141 represents the plan and elevation of a box on a scale of 1 cm. to 1 ft.

(i) Find the volume of the box.

(ii) Find the *total* area of its surface.

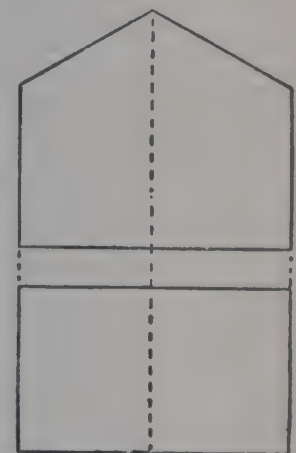


FIG. 141.

21. The diagram (Fig. 142), not drawn to scale, represents the plan of an estate of 5 acres. The measurements given are in inches. On what scale (inches to the mile) is it drawn? The dotted line PQ divides the estate in half; find AQ.

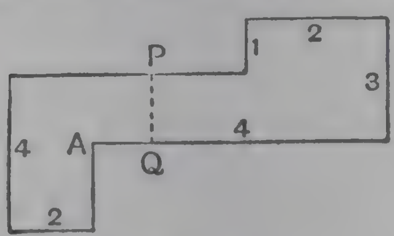


FIG. 142.

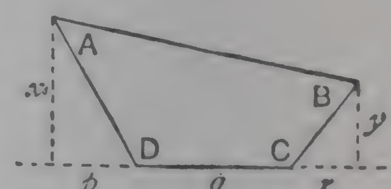


FIG. 143.

22. Find the area of ABCD (Fig. 143) in terms of x, y, p, q, r.

23. ABC is inscribed in a rectangle (Fig. 144); find the area of ABC in terms of p, q, r, s.

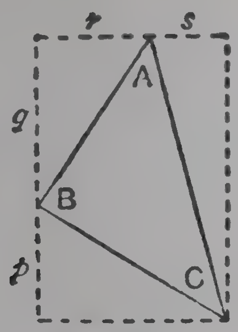


FIG. 144.

24. In Fig. 145, $\angle ABC = \angle BCD = 90°$. Find the length of the perpendicular from C to AD in terms of p, q, r.

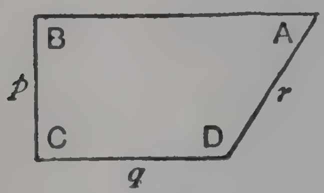


FIG. 145.

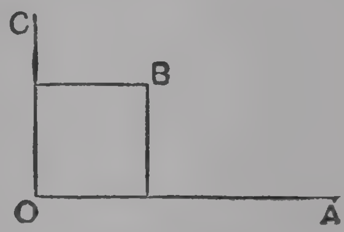


FIG. 146.

25. In Fig. 146, OB is a square, side 4″; OA=12″, OC=6″. Calculate areas of △OAB, △OBC, △AOC, and prove that ABC is a straight line.

26. P, Q are points on the sides AB, AD of the rectangle ABCD; AB=x, AD=y, PB=e, QD=f. Calculate area of PCQ in terms of e, f, x, y.

27. The area of a rhombus is 25 sq. cms., and one diagonal is half the other; calculate the length of each diagonal.

28. Find the area of the triangles whose vertices are:

(i) (2, 1); (2, 5); (4, 7).
(ii) (3, 2); (5, 4); (4, 8).
(iii) (1, 1); (5, 2); (6, 5).
(iv) (0, 0); (a, o); (b, c).
(v) (0, 0); (a, b); (c, d).

29. Find the area of the quadrilaterals whose vertices are:

(i) (0, 0); (3, 2); (1, 5); (0, 7).
(ii) (1, 3); (3, 2); (5, 5); (2, 7).

30. Find in acres the areas of the fields of which the following field-book measurements have been taken :

(1)

	YARDS.	
	to D	
	250	
to C 80	200	
	150	40 to E
to B 50	100	
	From A	

(2)

	YARDS.	
	to D	
	300	
	220	50 to E
to C 60	200	
to B 100	100	
	50	80 to F
	From A	

31. Find from the formula (page 106) the area of the triangles whose sides are (i) 5 cms., 6 cms., 7 cms.
(ii) 8″, 15″, 19″.

Find also in each case the greatest altitude.

32. The sides of a triangle are 7″, 8″, 10″. Calculate its shortest altitude.

---

33. AX, BY are altitudes of the triangle ABC ; if AC=2BC, prove AX=2BY.

34. ABC is a △ ; a line parallel to BC cuts AB, AC at P, Q ; prove △APC=△AQB.

35. Two lines AOB, COD intersect at O ; if AC is parallel to BD, prove △AOD=△BOC.

36. The diagonals AC, BD of ABCD are at right angles, prove that area of ABCD=$\frac{1}{2}$AC . BD.

37. The diagonals of the quad. ABCD cut at O ; if △AOB=△AOD, prove △DOC=△BOC.

38. In the triangles ABC, XYZ, AB=XY, BC=YZ, ∠ABC+∠XYZ =180°, prove △ABC=△XYZ.

39. P is any point on the median AD of △ABC ; prove △APB =△APC.

40. ABCD is a quadrilateral ; lines are drawn through A, C parallel to BD, and through B, D parallel to AC ; prove that the area of the parallelogram so obtained equals twice the area of ABCD.

41. ABCD is a parallelogram ; P is any point on AD ; prove that △PAB+△PCD=△PBC.

42. ABC is a straight line ; O is a point outside it ; prove $\frac{\triangle OAB}{\triangle OBC}=\frac{AB}{BC}$.

43. ABCD is a parallelogram; P is any point on BC; DQ is the perpendicular from D to AP; prove that the area of ABCD = DQ . AP.

44. ABCD is a parallelogram; P is any point on BD; prove △PAB = △PBC.

45. ABCD is a parallelogram; a line parallel to BD cuts BC, DC at P, Q; prove △ABP = △ADQ.

46. AOB is an angle; X is the mid-point of OB; Y is the mid-point of AX; prove △AOY = △BXY.

47. If in Fig. 147, AC is perpendicular to BD, prove area of ABCD = $\frac{1}{2}$AC . BD.

48. ABCD is a quadrilateral; a line through D parallel to AC meets BC produced at P; prove that △ABP = quad. ABCD.

49. ABCD is a quadrilateral; E, F are the mid-points of AB, CD; prove that

△ADE + △CBF = △BCE + △ADF.

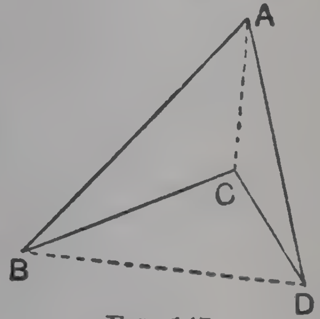


FIG. 147.

50. The diagonals of a quadrilateral divide it into four triangles of equal area; prove that it is a parallelogram.

51. X, Y are the mid-points of the sides AB, AC of △ABC; prove that △XBY = △XCY and deduce that XY is parallel to BC.

52. Two parallelograms ABCD, AXYZ of equal area have a common angle at A; X lies on AB; prove DX, YC are parallel.

53. The sides AB, BC of the parallelogram ABCD are produced to any points P, Q; prove △PCD = △QAD.

54. ABC is a △; D, E are the mid-points of BC, CA; Q is any point in AE; the line through A parallel to QD cuts BD at P; prove PQ bisects △ABC.

55. The medians BE, CF of △ABC intersect at G; prove that

△BGC = △BGA = △AGC.

56. In Fig. 148, the sides of △ABC are equal and parallel to the sides of △XYZ; prove that BAXY + ACZX = BCZY.

57. ABP, AQB are equivalent triangles on opposite sides of AB; prove AB bisects PQ.

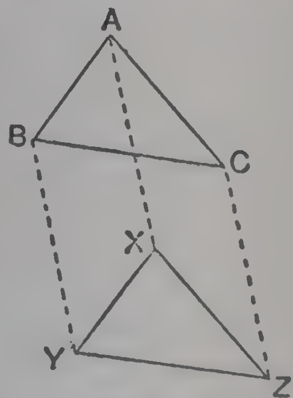


FIG. 148.

58. ABCD is a parallelogram; any line through A cuts DC at Y and BC produced at Z; prove △BCY = △DYZ.

59. In Fig. 149, PR is equal and parallel to AB; PQAT and

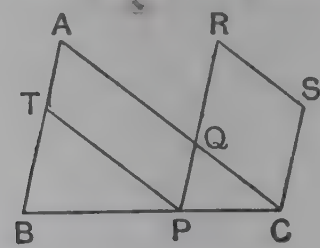


FIG. 149.

CQRS are parallelograms; prove they are equivalent.

60. BE, CF are medians of the triangle ABC and cut at G; prove

$$\triangle BGC = \text{quad. } AEGF.$$

61. In Fig. 150, APQR is a square; prove

$$\frac{1}{AP} = \frac{1}{AB} + \frac{1}{AC}.$$

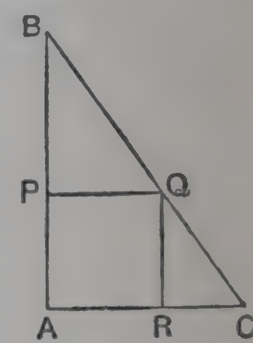


FIG. 150.

62. ABCD is a parallelogram; DC is produced to P; AP cuts BD at Q; prove

$$\triangle DQP - \triangle AQB = \triangle BCP.$$

63. In Fig. 151, ABCD is divided into four parallelograms; prove POSD=ROQB.

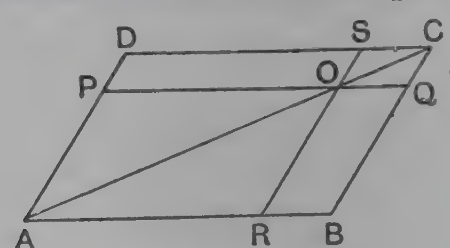


FIG. 151.

64. In Fig. 151, prove $\triangle APR + \triangle ASQ = \triangle ABD$.

65. ABCD is a parallelogram; AB is produced to E; P is any point within the angle CBE; prove

$$\triangle PAB + \triangle PBC = \triangle PBD.$$

66. In Fig. 152, ABCD is divided into four parallelograms, prove that

$$SOQD - BPOR = 2\triangle AOC.$$

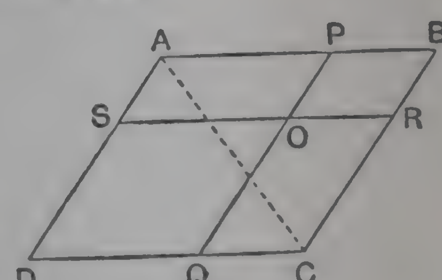


FIG. 152.

67. P is a variable point inside a fixed equilateral triangle ABC; PX, PY, PZ are the perpendiculars from P to BC, CA, AB; prove that PX+PY+PZ is constant.

CONSTRUCTION 12.

(1) Reduce a quadrilateral to a triangle of equal area.

Given a quadrilateral ABCD.

To construct a triangle equal in area to it.

Join BD.

Through C, draw CK parallel to DB to meet AB produced at K. Join DK.

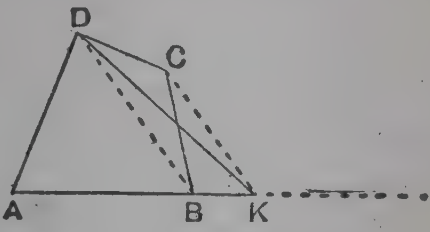


FIG. 153.

Then ADK is the required triangle.

Proof. The triangles BCD, BKD are on the same base BD and between the same parallels BD, KC.

$\therefore$ area of $\triangle$BCD = area of $\triangle$BKD.

Add to each $\triangle$ABD.

$\therefore$ area of quad. ABCD = area of $\triangle$AKD.

$\therefore$ AKD is the required triangle. Q.E.F.

In order to reduce a five-sided figure ABCDE to an equivalent triangle, proceed exactly as in Construction 12; we then obtain an equivalent four-sided figure AKDE. Repeat the process and we obtain an equivalent triangle.

The method can clearly be applied to any polygon.

EXERCISE XXXI.

1. Find the areas of the following figures, making any necessary constructions and measurements :

(i) $\triangle$ABC, given $b=5$, $c=4$, A$=90°$.

(ii) Rectangle ABCD, given AB$=7$, AC$=10$.

(iii) $\triangle$ABC, given $a=5$, $b=6$, $c=7$.

(iv) $\triangle$ABC, given $b=5$, $c=4$, B$=90°$.

(v) $\triangle$ABC, given $b=c=10$, $a=12$.

(vi) $\triangle$ABC, given $a=6$, B$=130°$, C$=20°$.

(vii) $\parallel$gram ABCD, given AB$=8$, AD$=6$, $\angle$ABC$=70°$.

(viii) A rhombus whose diagonals are 7, 8.

(ix) A trapezium ABCD, given AB$=5$, BC$=6$, CD$=9$, $\angle$BCD $=30°$, and AB parallel to DC.

(x) Quad. ABCD, given AB$=3$. BC$=5$. CD$=6$, DA$=4$, BD$=5$.

2. Draw a triangle whose sides are 5, 6, 8 cms., and obtain its area in three different ways.

3. Draw a triangle with sides 5, 6, 7 cm., and construct an isosceles triangle with base 6 cm. equal in area to it ; measure its sides.

4. Construct a parallelogram of area 21 sq. cm. such that one side is 6 cm., one angle is 50° ; measure the other side.

5. Construct a parallelogram of area 15 sq. cm. with sides 5 cm., 6 cm. ; measure its acute angle.

6. Draw a triangle with sides 4, 5, 6 cm., and construct a parallelogram equal in area to it and having one side equal to 4 cm. and one angle equal to 70° ; measure the other side.

7. Construct a rhombus each side of which is 5 cm. and of area 15 sq. cm. ; measure its acute angle.

8. Draw a parallelogram with sides 4 cm., 6 cm., and one angle 70° ; construct a parallelogram of equal area with sides 5 cm., 7 cm. ; measure its acute angle.

9. Construct a parallelogram of area 20 sq. cm., with one side 5 cm., and one diagonal 7 cm. ; measure the other side.

10. Draw a triangle with sides 5, 6, 8 cm., and construct a triangle of equal area with sides 5·5, 6·5 cm. ; measure the third side.

11. Draw a quadrilateral ABCD such that AB = 6 cm., BC = 5 cm., CD = 4 cm., ∠ABC = 110°, ∠BCD = 95°. Reduce it to an equivalent triangle with AB as base and its vertex on BC. Find its area.

12. Draw a figure like Fig. 154, and reduce it to an equivalent triangle having AB as base and its vertex on AD.

13. Draw a figure like Fig. 155, and reduce it to an equivalent triangle.

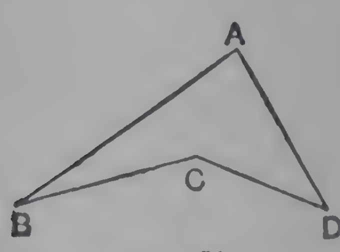


FIG. 154.

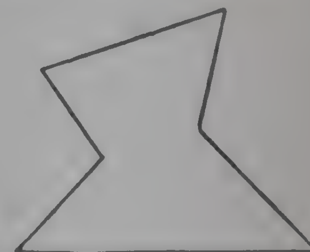


FIG. 155.

The next Exercise contains somewhat harder constructions, which may be reserved for a second reading.

EXERCISE XXXII.

1. Given a triangle ABC and a point K on BC; construct the line KP which bisects the area of the triangle. (D is mid-point of BC, DP is parallel to KA; join DA, see Fig. 156.)

2. Given a triangle ABC and a point K on BC; construct the line KP (Fig. 156) so that area KPB$=\frac{1}{3}$ area ABC.

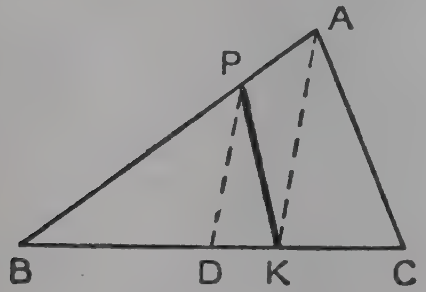


FIG. 156.

3. Given a triangle ABC and a point K on BC produced; construct a point P on BA so that area BPK=area BAC. (Note that area APK must equal area ACK.)

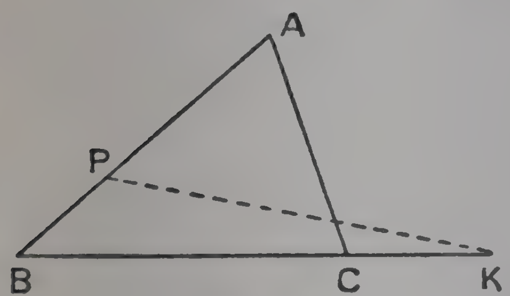


FIG. 157.

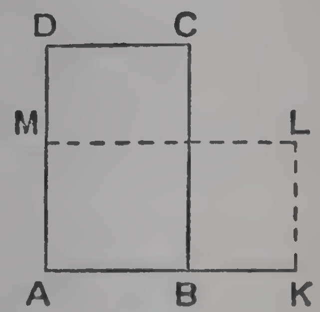


FIG. 158.

4. Given a rectangle ABCD and a point K on AB produced; construct a point M on AD so that the rectangle MAKL equals the rectangle ABCD in area. (Construct a triangle KAM equal to $\triangle$BAD, by using Ex. 3.)

5. Construct a rectangle equal in area to a given triangle and having one side of given length. (Use Ex. 4.)

6. Draw an equilateral triangle whose side is 3 inches long; construct a rectangle equal in area to it and having one side 2·5 inches long. Measure the other side.

7. Given a quadrilateral ABCD ; construct three lines AF, AG, AH dividing ABCD into 4 parts of equal area. (Reduce ABCD to

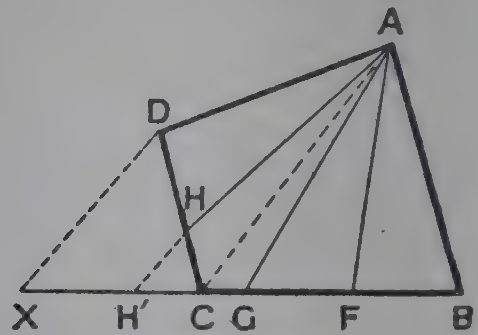


FIG. 159.

equivalent triangle ABX ; divide BX into four equal parts at F, G, H′ ; H′H is parallel to XD or AC.)

8. Show how to divide a quadrilateral into five parts of equal area by line through one of the corners.

9. ABCD is a given quadrilateral, E is a given point on AD. Show how to draw lines through E dividing the quadrilateral into three parts of equal area. (Consider the five-sided figure ABCDE.)

10. Given a parallelogram and a point O inside it, construct a line through O which divides ABCD into two parts of equal area.

11. Given a triangle ABC and a point D on BC such that $BD < \frac{2}{3}BC$; construct a point P on AC such that $\triangle DPC = \frac{2}{7}\triangle ABC$.

12. Given a parallelogram ABCD ; construct points P, Q on BC, CD such that AP, AQ divide ABCD into three parts of equal area.

13. Use the fact in Exercise XXX., No. 63, to construct a parallelogram equal in area to and equiangular to a given parallelogram and having one side of given length.

14. Given a parallelogram with sides 2·1 in., 2·9 in. and one angle 72° ; construct a rectangle of equal area with one side 2 inches long. Measure the other side.

Pythagoras' Theorem.

If you look at a floor tiled with equal tiles each in the shape of an isosceles right-angled triangle, you will probably see a connection between the area of the square on the hypotenuse and the sum of the areas of the squares on the other two sides

Fig. 160 represents a part of such a floor.

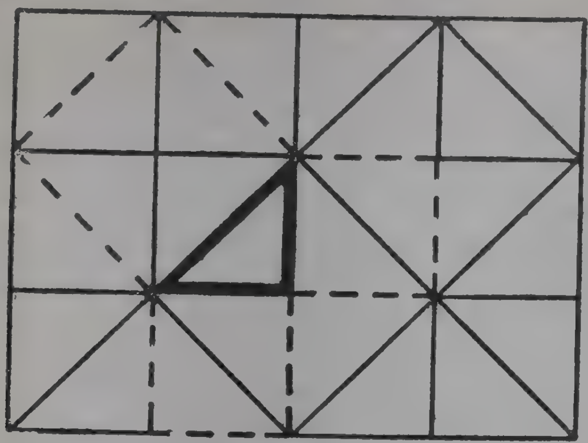


FIG. 160.

If the triangle is not isosceles, we may examine this connection in the following way:

Cut out a right-angled triangle from a sheet of paper, and then cut out three duplicates of it.

Suppose the hypotenuse is c in., and the other two sides are a in., b in.

Draw (or cut out) two equal squares of side $a+b$ in. and arrange the triangles on the top of each square, (i) as in Fig. 161 (1), (ii) as in Fig. 161 (2).

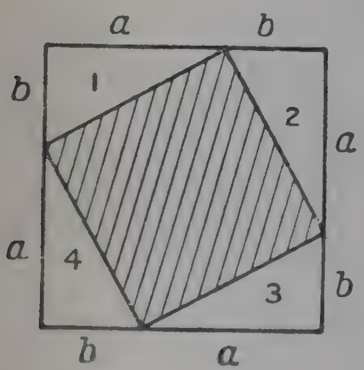


FIG. 161 (1).

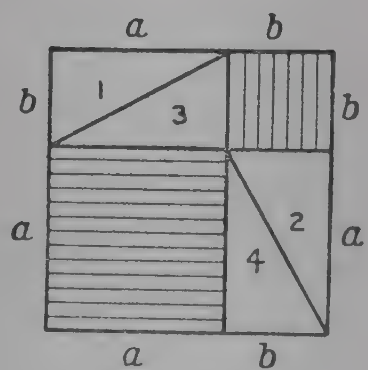


FIG. 161 (2).

What conclusion do you draw from this experiment?

Perigal's dissection.

Fig. 162 shows a method of dissection by which the squares on the two sides containing the right angle can be made to cover exactly the square on the hypotenuse, as a kind of jig-saw puzzle.

Draw a right-angled triangle ABC on thin cardboard or stiff paper and construct the three squares on its sides. Take the centre P of the square on AB and draw through P lines parallel

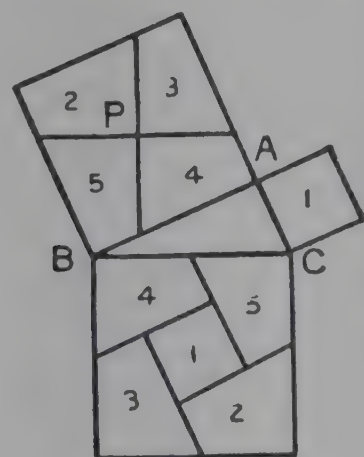


FIG. 162.

and perpendicular to BC. Cut out the squares on AB, AC and dissect the square on AB into the parts indicated. Arrange the pieces to cover the square on BC as shown in Fig. 162.

This theorem was discovered by Pythagoras, who lived in the sixth century B.C., but special cases of it were known to the Egyptians much earlier, at least by 1000 B.C., as their surveyors made use of the fact that 3, 4, 5 are the sides of a right-angled triangle.

The formal proof which comes next is due to Euclid, who wrote 300 years later the greatest text-book of all time, known as Euclid's *Elements*.

THEOREM 27. (Pythagoras' Theorem.)

In any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.

Given $\angle BAC$ is a right angle.

To prove the square on BC = the square on BA + the square on AC.

Let ABHK, ACMN, BCPQ be the squares on AB, AC, BC.

Join CH, AQ. Through A, draw AXY parallel to BQ, cutting BC, QP at X, Y.

Since $\angle BAC$ and $\angle BAK$ are right angles, KA and AC are in the same straight line.

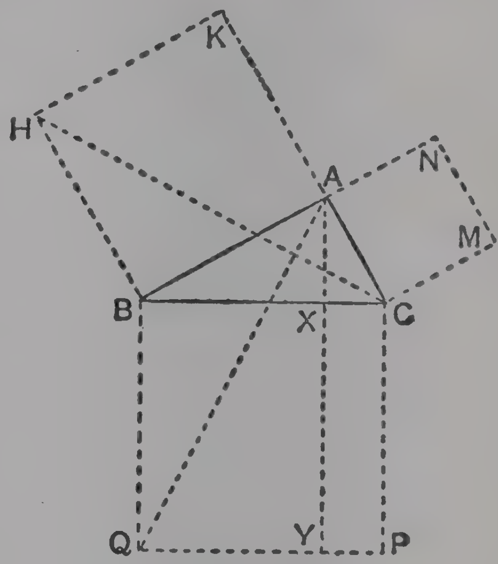


FIG. 163.

Again $\angle HBA = 90° = \angle QBC$.

Add to each $\angle ABC$, $\therefore$ $\angle HBC = \angle ABQ$.

In the $\triangle$s HBC, ABQ.

$$HB = AB, \text{ sides of square.}$$
$$CB = QB, \text{ sides of square.}$$
$$\angle HBC = \angle ABQ, \text{ proved.}$$
$$\therefore \triangle HBC \equiv \triangle ABQ.$$

Now $\triangle HBC$ and square HA are on the same base HB and between the same parallels HB, KAC;

$$\therefore \triangle HBC = \tfrac{1}{2} \text{ square } HA.$$

Also $\triangle ABQ$ and rectangle BQYX are on the same base BQ and between the same parallels BQ, AXY.

$$\therefore \triangle ABQ = \tfrac{1}{2} \text{ rect. } BQYX.$$
$$\therefore \text{square } HA = \text{rect. } BQYX.$$

Similarly, by joining AP, BM, it can be shown that square MA = rect. CPYX;

$$\therefore \text{square } HA + \text{square } MA = \text{rect. } BQYX + \text{rect. } CPYX$$
$$= \text{square } BP. \qquad \text{Q.E.D.}$$

THEOREM 28.

If the square on one side of a triangle is equal to the sum of the squares on the other sides, then the angle contained by these sides is a right angle.

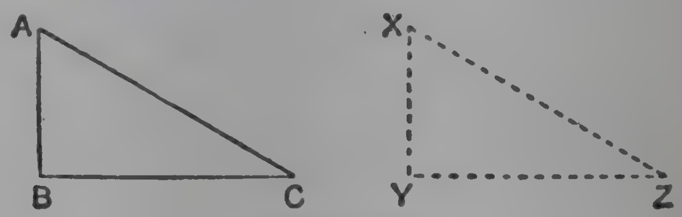


FIG. 164.

Given $AB^2 + BC^2 = AC^2$.

To prove $\angle ABC = 90°$.

Construct a triangle XYZ such that $XY = AB$, $YZ = BC$, $\angle XYZ = 90°$.

Since $\angle XYZ = 90°$, $XZ^2 = XY^2 + YZ^2$.

But $XY = AB$ and $YZ = BC$.

$\therefore\ XZ^2 = AB^2 + BC^2 = AC^2$ given.

$\therefore\ XZ = AC$.

$\therefore$ in the $\triangle$s ABC, XYZ

$AB = XY$, constr.

$BC = YZ$, constr.

$AC = XZ$, proved.

$\therefore\ \triangle ABC \equiv \triangle XYZ$.

$\therefore\ \angle ABC = \angle XYZ$.

But $\angle XYZ = 90°$ constr.

$\therefore\ \angle ABC = 90°$. Q.E.D.

EXERCISE XXXIII.

1. In Fig. 163, $AB = 5''$, $AC = 12''$; calculate BC, BX.
2. In Fig. 163, $AC = 6''$, $BC = 10''$; calculate AB, CX.
3. In Fig. 163, $AB = 7''$, $BC = 9''$; calculate AC, BX.
4. In $\triangle ABC$, $AB = AC = 9''$, $BC = 8''$; calculate area of $\triangle ABC$.
5. In $\triangle ABC$, $AB = AC = 13''$, $BC = 10''$; calculate the length of the altitude BE.
6. Find the side of a rhombus whose diagonals are 6, 10 cm.

7. A kite at P, flown by a boy at Q, is vertically above a point R on the same level as Q; if PQ=505′, QR=456′, find the height of the kite.

8. In △ABC, AC=3″, AB=8″, ∠ACB=90°; find the length of the median AD.

9. AD is an altitude of △ABC; AD=2″, BD=1″, DC=4″; prove ∠BAC=90°.

10. ABCD is a parallelogram; AC=13″, BD=5″, ∠ABD=90°; calculate area of ABCD.

11. A gun, whose effective range is 9000 yards, is 5000 yards from a straight railway; what length of the railway is commanded by the gun?

12. The lower end of a 20-foot ladder is 10 feet from a wall; how high up the wall does the ladder reach? How much closer must it be put to reach one foot higher?

13. An aeroplane heads due North at 120 miles an hour in an east wind blowing at 40 miles an hour; find the distance travelled in ten minutes.

14. A ship is steaming at 15 knots and heading N.W.; there is a 6-knot current setting N.E.; how far will she travel in one hour?

15. AB, AC are two roads meeting at right angles; AB=110 yards, AC=200 yards; P starts from B and walks towards A at 3 miles an hour; at the same moment Q starts from C and walks towards A at 4 miles an hour. How far has P walked before he is within 130 yards of Q?

16. Find the distance between the points (1, 2), (5, 4).

17. Prove that the points (5, 11), (6, 10), (7, 7) lie on a circle whose centre is (2, 7); and find its radius.

18. The parallel sides of an isosceles trapezium are 5″, 11″, and its area is 32 sq. inches; find the lengths of the other sides.

19. In △ABC, ∠ABC=90°, ∠ACB=60°, AC=8″; find AB.

20. In △ABC, ∠ABC=90°, ∠ACB=60°, AB=5″; find BC.

21. In Fig. 165, AB=2″, BC=4″, CD=1″; find AD.

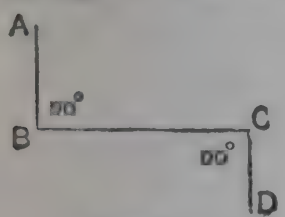


FIG. 165.

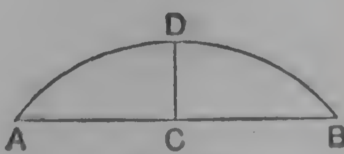


FIG. 166.

22. In Fig. 166, AC=CB=12″, CD=8″, ∠ACD=90°; find radius of circular arc.

23. Prove that the triangle whose sides are x^2+y^2, x^2-y^2, $2xy$ is right-angled.

24. AD is an altitude of the triangle ABC; BD$=x^2$, DC$=y^2$, AD$=xy$; prove that $\angle$BAC$=90°$.

25. AD is an altitude of $\triangle$ABC, $\angle$BAC$=90°$; AD$=4''$, CD$=3''$; calculate AB.

26. AD, BC are two vertical poles, D and C being the ends on the ground, which is level; AC$=12'$, AB$=10'$, BC$=3'$; calculate AD.

27. In $\triangle$ABC, AB$=$AC$=10''$, BC$=16''$; find the radius of the circle which passes through A, B, C.

28. In $\triangle$ABC, AB$=4''$, BC$=5''$, $\angle$ABC$=45°$; calculate AC.

29. In $\triangle$ABC, AB$=8''$, BC$=3''$, $\angle$ABC$=60°$; calculate AC.

30. The slant side of a right circular cone is $10''$, and the diameter of its base is $8''$; find its height.

31. Find the diagonal of a cube whose edge is $5''$.

32. A room is 20 feet long, 16 feet wide, 8 feet high; find the length of a diagonal.

33. A piece of wire is bent into three parts AB, BC, CD each of the outer parts being at right angles to the plane containing the other two; AB$=12''$, BC$=6''$, CD$=12''$; find the distance of A from D.

34. A hollow sphere, radius $8''$, is filled with water until the surface of the water is within $3''$ of the top. Find the radius of the circle formed by the water-surface.

35. A pyramid of height $8''$ stands on a square base each edge of which is $1'$. Find the area of the sides and the length of an edge.

36. ABCD is a rectangle; AB$=6''$, BC$=8''$; it is folded about BD so that the planes of the two parts are at right angles. Find the new distance of A from C.

---

37. AD is an altitude of the equilateral triangle ABC; prove that AD$^2=\frac{3}{4}$BC^2.

38. In $\triangle$ABC, CD is an altitude; prove

$$AC^2+BD^2=BC^2+AD^2.$$

39. ABN, PQN are two perpendicular lines; prove that

$$PA^2+QB^2=PB^2+QA^2.$$

40. The diagonals AC, BD of the quadrilateral ABCD are at right angles; prove that AB$^2+$CD$^2=$AD$^2+$BC^2.

41. If in the quadrilateral ABCD, $\angle ABC = \angle ADC = 90°$; prove that $AB^2 - AD^2 = CD^2 - CB^2$.

42. P is a point inside a rectangle ABCD; prove that $PA^2 + PC^2 = PB^2 + PD^2$. Is this true if P is outside ABCD?

43. In $\triangle ABC$, $\angle BAC = 90°$; H, K are the mid-points of AB, AC; prove that $BK^2 + CH^2 = \frac{5}{4}BC^2$.

44. ABCD is a rhombus; prove that $AC^2 + BD^2 = 2AB^2 + 2BC^2$.

45. In the quadrilateral ABCD, $\angle ACB = \angle ADB = 90°$; AH, BK are drawn perpendicular to CD; prove $DH^2 + DK^2 = CH^2 + CK^2$.

46. AD is an altitude of $\triangle ABC$; P, Q are points on AD produced such that PD = AB and QD = AC; prove BQ = CP.

47. In $\triangle ABC$, $\angle BAC = 90°$; AD is an altitude; prove $AD = \dfrac{AB \cdot AC}{BC}$.

48. In $\triangle ABC$, $\angle BAC = 90°$; AX is an altitude; use Fig. 163, and the proof of Pythagoras' theorem to show that $BA^2 = BX \cdot BC$; and deduce that $\dfrac{AB^2}{AC^2} = \dfrac{BX}{CX}$.

49. In $\triangle ABC$, $\angle BAC = 90°$; AD is an altitude; prove that $AD^2 = BD \cdot DC$.

50. ABC is an equilateral triangle; D is a point on BC such that BC = 3BD; prove $AD^2 = \frac{7}{9}AB^2$.

51. ABC is an equilateral triangle; D, E are the mid-points of BC, CD; prove $AE^2 = 13EC^2$.

52. In the $\triangle ABC$, AB = AC = 2BC; BE is an altitude; prove that AE = 7EC.

53. O is any point inside $\triangle ABC$; OP, OQ, OR are the perpendiculars to BC, CA, AB; prove $BP^2 + CQ^2 + AR^2 = PC^2 + QA^2 + RB^2$.

54. Fig. 167 shows a square of side $a+b$ divided up; use area formulæ to prove Pythagoras' theorem $a^2 + b^2 = c^2$.

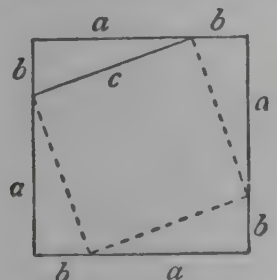


FIG. 167.

55. ABC is a straight line; ABXY, BCPQ are squares on the same side of AC; prove $PX^2 + CY^2 = 3(AB^2 + BC^2)$.

Constructions depending on Pythagoras' Theorem.

EXERCISE XXXIV.

1. Construct a line of length (i) $\sqrt{2}$, (ii) $\sqrt{5}$ inches.

2. Construct a square of area (i) 13 sq. in., (ii) 7 sq. in.

3. Given a square, show how to construct a square of twice the area.

4. Given two squares, show how to construct a square equal in area to (i) the sum of the two squares, (ii) the difference of the two squares.

5. Construct a square one-third of the area of a given square.

6. In Fig. 163, use the facts that (i) area BAKH = area BXYQ, (ii) circle on BC as diameter passes through A, to construct a square equal in area to a given rectangle.

7. Given a line AB, construct on AB a point P such that the sum of the squares on AP, PB is equal to the area of a given square. When is this impossible? (Draw a line BC making $\angle ABC = 45°$, if Q is any point on BC and QN the perpendicular from Q to AB, what can you say about QN and QA^2?)

8. Given a line AB, construct on AB a point P such that $AP^2 = 2PB^2$.

SECTION III.

LOCI.

If we look at the tip of the seconds hand of a watch we see that it occupies a series of positions in the course of each minute : the tip of the seconds hand traces out a curve which is called its **locus**. If a point moves about, subject to some fixed condition, the path traced out by the point is called the **locus of the point**, and we may regard the locus as the aggregate of all possible positions of the point, subject to the given law. When we state that the locus of a point, which can move subject to some given condition, is a certain curve, two complementary ideas are involved.

(i) Every point on the curve satisfies the given condition.

(ii) Every point which satisfies the given condition lies on the curve.

It may happen, however, that the conditions of the problem prevent the point from describing the whole of a curve : in this case it should always be stated what *part* of the curve forms the locus. Suppose, for example, a door is capable of being opened through an angle of 110° but no more, what is the locus of the door-handle, or more precisely a small mark on the handle ? The locus in this case is an arc of a circle, the arc being such that it subtends an angle of 110° at its centre.

The following example of a restricted curve is instructive :

A, B are two fixed points, and a variable point P moves in a fixed plane through A, B, so that $\angle APB = 60°$. Find, experimentally, the locus of P.

Stick two pins into the paper at A and B, perpendicular to the paper, and slide your set square between the pins so that the arms of the angle 60° of the set-square pass through A, B.

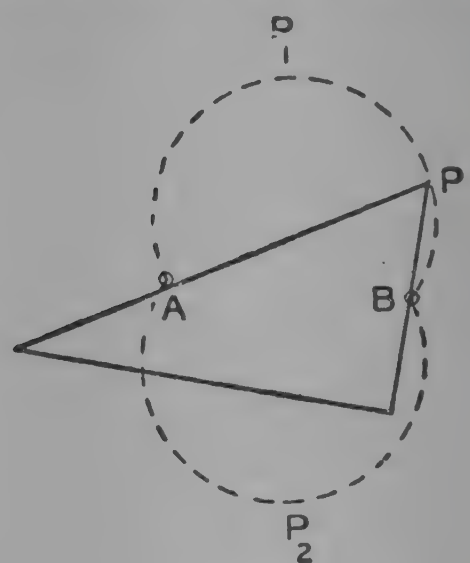


FIG. 168.

Prick in the paper the various positions of the vertex of the angle.

We see that the locus consists of two arcs of two distinct circles, AP_1B and AP_2B, not the whole circumference of one circle. [Compare in particular Ex. xxxv., No. 19-24].

Further, if P could move anywhere in space, its locus would be a surface obtained by revolving the arc AP_1B about the line AB as axis through 4 right angles.

Note.—The formal proof of this locus depends on Theorem 37. For the present, it should be regarded as an experimental result.

EXERCISE XXXV.

What are the loci described in Ex. 1-9 ?

1. A child in a swing.

2. The centre of the wheel of an engine running along a straight railway line.

3. A donkey tethered to a peg, if it keeps its chain taut.

4. A passenger sitting in a train which is running round a circular track.

5. A child on a see-saw.

6. The top of a child's head if he slides down stairs on a tea-tray.

7. The tip of the pendulum of a clock.

8. The centre of a marble which rolls about inside a spherical bowl.

9. A man who walks about so that he remains at equal distances from two trees.

10. A penny is held flat on the table and another penny also flat on the table is made to roll round it ; what is the locus of the centre of the moving penny ?

11. A circular disc is pivoted about a point on its rim ; what is the locus of another marked point on the rim ?

12. A dog is chained to a low straight rail 10 feet long by a chain 6 feet long, which can slide along the rail. What is his locus if he keeps the chain taut ?

13. ABC is an angle iron, the parts AB, BC being at right angles it rotates about A ; what is the locus of C ?

14. A cart is moving in a straight line on level ground and comes to a raised obstacle with a level top, which it climbs and passes over and returns to level ground on the other side. What is the locus of the centre of one of the wheels ?

15. A thin walking-stick is 3 feet long ; a point P moves in space so that its shortest distance from the stick is 2 feet. What is the complete locus of P ?

16. A water-cart with pump attached can be wheeled to any part of a rectangular lawn surrounded by flower beds ; what is the boundary of the portion that can be watered ?

17. AB is a long string with a weight attached to B; the end A is free to slide on the rim of a fixed vertical circular ring and AB itself remains vertical. What is the locus of B?

18. What is your locus if you walk along the floor of a square room so that the sum of your distances from two adjacent walls is (i) always equal to the length of the room, (ii) always equal to one-half of the length of the room?

19. ABCD is a fixed rhombus: a point P moves in the plane of ABCD so that the angle APB equals the angle APD. Discuss whether the diagonals and the diagonals produced belong to the locus. Do any points not on the diagonals belong to the locus?

20. What is the locus of a point which moves inside a large triangle so that it is always two inches from some side and never less than two inches from any side.

21. ABC is an equilateral triangle of side one inch; a point P moves in the plane of ABC so that it is always one inch away from some one of the three points A, B, C and never less than one inch from the other two points. What is the locus of P?

22. A cubical packing-case, section ABCD, rests with AB on the floor: it is rolled, without slipping, on to its side BC, and then rolled over again on to CD, and then on to DA. What is the locus of A?

23. Repeat Ex. 22 for a case whose section is an equilateral triangle ABC.

24. (i) Take a fixed point A and a fixed line CD of indefinite length; a point P moves so that its distance from A is equal to its shortest distance from CD. Construct a number of positions of P and so draw a free-hand curve to represent the locus. (It is called a parabola.)

(ii) Take a fixed line AB of finite length and a fixed line CD of indefinite length in a direction perpendicular to AB, but not cutting it between A and B; a point P moves so that its shortest distance from the line CD, produced if necessary, is equal to its distance from the nearest point of the finite line AB (*i.e.* not produced beyond A or B). Describe the locus of P.

Intersection of Loci.

If the position of a point is given by two distinct conditions, it may be possible to trace the two corresponding loci, and so fix its position from the intersection of these lines or curves.

EXERCISE XXXVI.

1. ABC is an equilateral triangle of side three inches, what is the locus of all points two inches from A? What is the locus of all points one inch from BC? Construct the position (or positions of) a point P two inches from A and one inch from the line BC.

2. With the data of Ex. 1, find a point on AC which is 2·8 inches from B.

3. With the data of Ex. 1, find a point on BC whose distance from AB is two inches.

4. A is a point whose distance from a line BC is 7 cm.: a circle centre A radius 5 cm. is drawn. Construct a point Q on the circle such that if QR makes an angle of 45° with BC, and cuts BC at R, then QR=4 cm.

5. ABC is an equilateral triangle of side 6 cm. Construct a point P such that its perpendicular distances from AB, AC are 2 cm., 3 cm. respectively.

6. ABC is a triangle such that AB=5 cm., BC=6 cm., CA=7 cm. Construct a point on AC which is equidistant from AB and BC.

7. With the data of Ex. 6, construct a point P such that PA=PB and PC=5 cm.

8. With the data of Ex. 6, construct a point P whose distance from BC is 3 cm., and such that it is equidistant from A, C,

9. With the data of Ex. 6, construct a point P such that it is 2 cm. from AB, and is equidistant from AC and BC.

10. With the data of Ex. 6, construct the centre of a circle, if the centre lies on AC and the circle passes through B and C.

11. Draw a triangle of any shape, and construct the position of a point equidistant from the three corners.

12. Draw a triangle of any shape, and construct the position (or positions) of a point equidistant from the three sides, produced if necessary.

THEOREM 29.

The locus of a point, which is equidistant from two given points, is the perpendicular bisector of the straight line joining the given points.

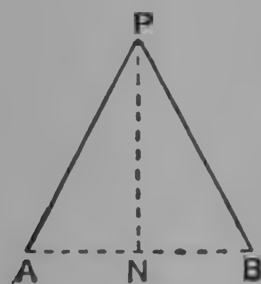


FIG. 169.

Given two fixed points A, B and any position of a point P which moves so that PA = PB.

To prove that P lies on the perpendicular bisector of AB.

Bisect AB at N. Join PN.

In the $\triangle$s ANP, BNP, AN = BN, constr.

AP = BP, given. PN is common.

$\therefore$ $\triangle ANP \equiv \triangle BNP$.

$\therefore$ $\angle ANP = \angle BNP$.

But these are adjacent angles, $\therefore$ each is a right angle.

$\therefore$ PN is perpendicular to AB and bisects it.

$\therefore$ P lies on the perpendicular bisector of AB.

Conversely, any point P on the perpendicular bisector of AB is equidistant from A and B.

For in the $\triangle$s ANP, BNP, AN = BN given,

PN is common,

$\angle ANP = 90° = \angle BNP$, given,

$\therefore$ $\triangle ANP \equiv \triangle BNP$ and AP = BP.

Q.E.D.

THEOREM 30.

The locus of a point which is equidistant from two given intersecting straight lines is the pair of lines which bisect the angles between the given lines.

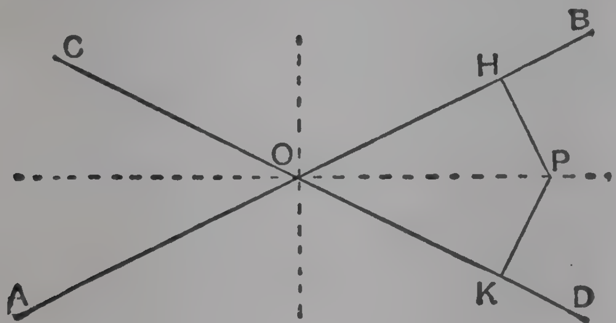


FIG. 170.

Given two fixed lines AOB, COD and any position of a point P which moves so that the perpendiculars PH, PK from P to AOB, COD are equal.

To prove P lies on one of the two lines bisecting the angles BOC, BOD.

Suppose P is situated in the angle BOD.

In the *right-angled* triangles PHO, PKO,

PH = PK, given.

PO is the common hypotenuse.

$\therefore \triangle PHO \equiv \triangle PKO.$ $\quad \therefore \angle POH = \angle POK.$

$\therefore$ P lies on the line bisecting the angle BOD.

In the same way if P is situated in either of the angles BOC, COA, AOD, it lies on the bisectors of these angles.

Conversely, any point P on the line bisecting either of the angles between the lines AOB, COD is equidistant from these lines.

For in the $\triangle$s PHO, PKO, $\angle POH = \angle POK$ given, $\angle OHP = 90°$ $= \angle OKP$, OP is the common hypotenuse, $\therefore \triangle PHO \equiv \triangle PKO$ and PH = PK.

Q.E.D.

Definition.

If the straight line AB is the perpendicular bisector of the line joining two points P, P′, then P′ is called the image or reflection of P in AB.

EXERCISE XXXVII.

1. A variable point is at a given distance from a given point, what is its locus, (i) in a plane, (ii) in space ?

2. A variable point is at a given distance from a given line, what is its locus, (i) in a plane, (ii) in space ?

3. A variable circle passes through two fixed points, what is the locus of its centre, (i) in a given plane through the two points, (ii) in space ?

4. A variable circle of given radius passes through a fixed point, what is the locus of its centre, (i) in a plane, (ii) in space ?

5. A, B are fixed points ; APB is a triangle of given area ; what is the locus of P ?

6. A variable chord of a fixed circle is of given length, what is the locus of its mid-point ?

7. P is a variable point on a given line ; O is a fixed point outside the line ; find the locus of the mid-point of OP.

8. A, B are fixed points ; PAQB is a variable parallelogram of given area ; find the complete locus of P.

9. ABC is a given triangle ; BAPQ, CBQR are variable parallelograms ; if P moves on a fixed circle, centre A, find the locus of R.

10. A, B are fixed points ; ABPQ is a variable parallelogram ; if AP is of given length, find the locus of Q.

11. PQR is a variable triangle ; the mid-points of PQ and PR are fixed and QR passes through a fixed point ; find the locus of P.

12. A, B are fixed points ; P is a variable point such that $PA^2 - PB^2$ is constant ; prove that the locus of P is a straight line perpendicular to AB.

13. A, B, C, D are fixed points ; P is a variable point, such that the sum of the areas of the triangles PAB, PCD is constant ; prove that the locus of P is a straight line. (Let AB produced cut CD produced at O, from OB, OD cut off OH, OK equal to AB, CD ; the line is parallel to HK.)

14. If P′ is the image of P in the line AB, prove that AP=AP′.

15. A variable line OQ passes through a fixed point O ; A is another fixed point ; find the locus of the image of A in OQ.

16. A, B are two points on the same side of a line CD ; A′ is the image of A in CD ; A′ B cuts CD at O ; prove that (i) AO and OB

make equal angles with CD; (ii) if P is any other point on CD, AP+PB>AO+OB.

17. AH, BK are the perpendiculars from A, B to XY. AH=5″, BK=7″, HK=16″; what is the least value of AP+PB ?

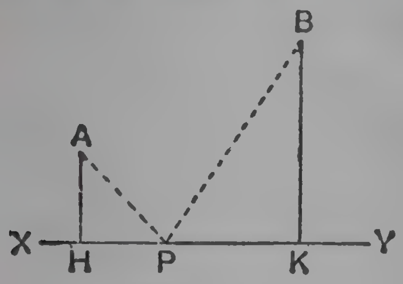


FIG. 171.

18. A, B are fixed points on opposite sides of a fixed line CD; find the point P on CD for which PA ~ PB has its greatest value.

19. How many images are formed when a candle is placed between two plane mirrors inclined to each other at an angle of (i) 90°, (ii) 60° ?

The remaining examples involve angle properties of a circle.

20. A variable circle touches a fixed line at a fixed point, what is the locus of its centre ?

21. A variable circle touches two fixed lines, what is the locus of its centre ?

22. A variable circle of given radius touches a fixed circle, what is the locus of its centre ?

23. A variable circle touches two fixed concentric circles, what is the locus of its centre ?

24. A variable circle of given radius touches a given line, what is the locus of its centre ?

25. PQR is a variable triangle; ∠QPR=90°, PQ and PR pass through fixed points; what is the locus of P ?

26. Given the base and vertical angle of a triangle, find the locus of its vertex.

27. A is a fixed point on a fixed circle; AP is a variable chord; find the locus of the mid-point of AP.

28. A variable chord PQ of a given circle passes through a fixed point; find the locus of the mid-point of PQ.

29. The extremities of a line of given length move along two fixed perpendicular lines; find the locus of its mid-point.

REVISION PAPERS.

SECTIONS I.-III.

19.

1. If in Fig. 46, p. 35, $x+y=2z$, find z.

2. ABC is a triangle, right-angled at A ; AD is the perpendicular from A to BC ; prove that $\angle ABC=\angle DAC$.

3. If in Fig. 144, p. 109, $p=3''$, $q=4''$, $r=2''$, $s=1''$, find the area of $\triangle ABC$.

4. P, Q, R are points on the sides BC, CA, AB of a triangle ABC, such that PQ is parallel to AB and QR is parallel to BC ; prove that $\triangle ABP=\triangle ACR$.

20.

1. (i) For what reason does a difference of 15° in longitude cause a difference in local time of 1 hour ?

(ii) When it is noon at Greenwich (long. 0°), what is the local time at (*a*) Boston, long. 71° W., (*b*) Bombay, long. 73° E. ?

2. P is a point on the side AB of $\triangle ABC$ such that $AP=PC=CB$. If CP bisects $\angle ACB$, calculate $\angle BAC$.

3. If in Fig. 139, p. 107, $AD=\frac{1}{2}AB$, prove that $AP=2AQ$.

4. The vertices of a triangle are (2, 0), (0, 5), (3, 7), the unit on each axis being one inch. Calculate the area of the triangle.

21.

1. K is a point inside $\triangle ABC$, such that KB, KC bisect $\angle ABC$, $\angle ACB$. If $\angle BKC=112°$, calculate $\angle BAC$.

2. ABC is a triangle ; P, R, T are points on AB and Q, S are points on AC such that $AP=PQ=QR=RS=ST$; prove that $\angle CST=5\angle AQP$.

3. The perimeter of a right-angled triangle is 2S inches and the length of the hypotenuse is C inches ; prove that the area of the triangle is $S(S-C)$ sq. inches.

4. ABCD is a quadrilateral ; if $AB=BC=2CD$ and $\angle ABD=\angle BCD=90°$, prove that $AD=3CD$.

22.

1. Prove that the sum of the interior angles of an 11-sided convex polygon is three times that of a convex pentagon.

2. ABCD is a parallelogram and O is the mid-point of AB; if DC=2AD, prove $\angle DOC = 90°$.

3. If in Fig. 145, p. 109, $p=12''$, $q=11''$, $r=13''$, calculate the length of AC.

4. D is any point on the base BC of $\triangle ABC$; DE, DF are the perpendiculars from D to AB, AC; prove that EF<BC.

23.

1. OA, OB, OC are three straight lines in order such that $\angle AOB = x°$, $\angle BOC = y°$ and $x > y$; OK bisects $\angle AOC$. Find $\angle KOB$ in terms of x, y if the $\angle$s AOB, BOC are (i) both acute, (ii) both obtuse.

2. ABC is an equilateral triangle; K is a point on BC such that $\angle CAK = 3\angle KAB$; a line KL is drawn perpendicular to BC to meet AB at L; prove that AL=LK.

3. ABC is an equilateral triangle of side 6″; P is a point on BC such that $BP = \frac{1}{3}BC$; calculate the length of AP.

4. In $\triangle ABC$, BC>AC and $\angle ACB = 90°$; Q is a point on BC such that $\angle QAB = \angle QBA$; prove that $AB^2 = 2AQ . BC$.

24.

1. What is the least number of measurements that must be taken to make an accurate copy of (i) a quadrilateral, (ii) a pentagon, (iii) an octagon, (iv) an n-sided polygon ?

2. OX, OY are two plane mirrors; a ray of light starting from A moves along AP, is then reflected along PQ and finally reflected along QB. An optical law states that PA, PQ make equal angles with OX and that QP, QB make equal angles with OY. Prove that the angle between the initial ray AP and the final ray QB equals $2\angle XOY$.

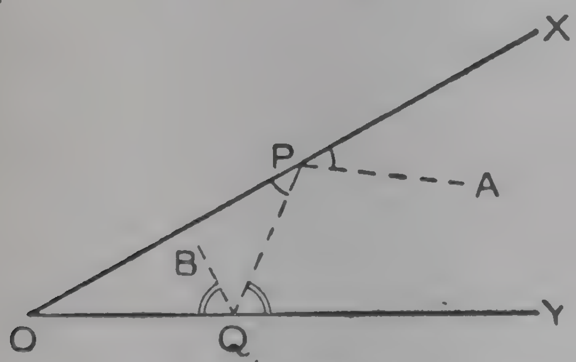


FIG. 172.

3. ABCD is a quadrilateral, O is the mid-point of BD; prove that $\triangle ADC \sim \triangle ABC = 2\triangle AOC$.

4. If in Fig. 171, p. 133, AH $=3''$, HK $=10''$, KB $=8''$ and AP $=\frac{1}{2}$PB, calculate the lengths of HP, AB. Prove also that $\angle$APB $=90°$.

25.

1. AD, BE are altitudes of $\triangle$ABC; BC $=5$ cm., CA $=6$ cm., AD $=4{\cdot}5$ cm.; find BE.

2. ABC is an equilateral triangle; P, Q are points on BC, CA such that BP $=$ CQ; AP cuts BQ at R; prove $\angle$ARB $=120°$.

3. P is a variable point on a circle, centre O, radius a; C is a fixed point at a distance b from O; find the greatest and least possible lengths of CP.

4. ABCD is a quadrilateral; if $\triangle$ACD $=\triangle$BCD, prove $\triangle$ABC $=\triangle$ABD.

26.

1. Find in terms of x, y, z the area of Fig. 173.

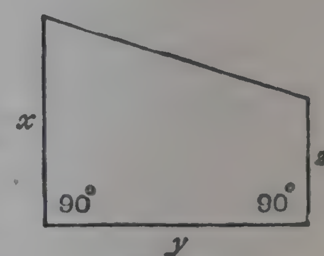


FIG. 173.

2. In $\triangle$ABC, AB $=$ AC; a line PQR cuts AC produced, AB, BC at R, P, Q; if PQ $=$ QR, prove AP $+$ AR $=2$AC.

3. The diagonals of the quadrilateral ABCD cut at O; if $\triangle$AOD $=\triangle$BOC, prove $\triangle$s AOB, COD are equiangular.

4. In $\triangle$ABC, $\angle$BAC $=90°$, AB $=5$ cm., AC $=8$ cm.; find the area of the triangle and the length of its altitude AD.

27.

1. Find in sq. cm. the area, making any construction and measurements, of Fig. 174.

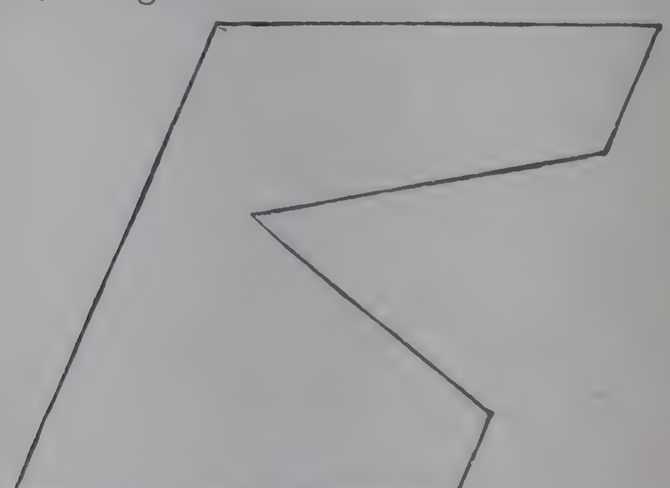


FIG. 174.

2. ABCDE is a regular pentagon; BD cuts CE at P; prove BP=BA.

3. The hypotenuse of a right-angled triangle is $x^2+\frac{1}{x^2}$ inches long, and one of the other sides is $x^2-\frac{1}{x^2}$ inches. Find the third side.

4. The side BC of the parallelogram ABCD is produced to any point K; prove $\triangle$ABK=quad. ACKD.

28.

1. ABCD is a parallelogram of area 24 sq. cm.; its diagonals intersect at O; AB=4·5 cm.; find the distance of O from CD.

2. In $\triangle$ABC, $\angle$BAC=90°; BDEC is a square outside $\triangle$ABC; DX is the perpendicular from D to AC; prove DX=AB+AC.

3. BE, CF are altitudes of $\triangle$ABC; prove $\frac{AB}{AC}=\frac{BE}{CF}$.

4. AD is an altitude of $\triangle$ABC; AB=7, AC=5, BC=8; if BD=x, DC=y, prove $x^2-y^2=24$, and find x, y; find also the area of $\triangle$ABC.

29.

1. In Fig. 175, ABCD is a quadrilateral inscribed in a rectangle find the area of ABCD in terms of p, q, r, s, x, y.

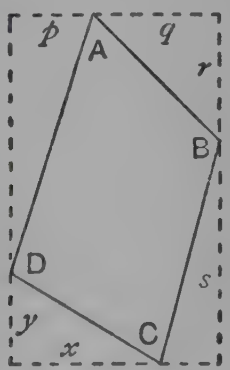


Fig. 175.

2. In $\triangle$ABC, $\angle$BAC=90°; P, Q are points on BC such that CA=CP and BA=BQ; prove $\angle$PAQ=45°.

3. ABCD is a quadrilateral; $\angle$ABC=$\angle$ADC=90°; AP, AQ are drawn parallel to CD, CB, cutting CB, CD at P, Q; prove QA . AB =PA . AD. (Use area formulæ.)

4. What is the length of the diagonal of a box whose sides are 3″, 4″, 12″?

30.

1. AD, BE, CF are the altitudes of $\triangle$ABC; AB$=5x$ cm., BC$=6x$ cm., CA$=3x$ cm., AD$=7\cdot5$ cm.; find BE, CF.

2. The base BC of the triangle ABC is produced to D; the lines bisecting $\angle$s ABD, ACD meet at P; a line through P parallel to BC cuts AB, AC at Q, R; prove QR$=$BQ$\sim$CR.

3. ABCD is a rhombus; P, Q are points on BC, CD such that BP$=$CQ; AP cuts BQ at O; prove $\triangle$AOB$=$quad. OPCQ.

4. In Fig. 176, AB$=2''$, BC$=4''$, CD$=1''$; if PD$^2=2$PA^2, find PB.

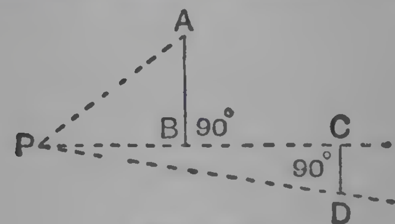


FIG. 176.

31.

1. Soundings are taken at intervals of 4 feet across a river 40 feet wide, starting 4 feet from one bank, and the following depths in feet are obtained in order: 6·6, 9·3, 9·9, 8·2, 8·4, 10·2, 10·5, 7·8, 4·5; find approximately the area of the river's cross-section.

2. In the $\triangle$ABC, AB$=$BC and $\angle$ABC$=90°$; the bisector of $\angle$BAC cuts BC at D; prove AB$+$BD$=$AC.

3. ABCD is a parallelogram; P is the mid-point of AD; AB is produced to Q so that AB$=$BQ; prove ABCD$=2\triangle$PQD.

4. In $\triangle$ABC, $\angle$BAC$=90°$; P is the mid-point of AC; PN is drawn perpendicular to BC; prove BN$^2=$BA$^2+$CN^2.

32.

1. ABCD is a parallelogram; AB$=4x$ cm., BC$=5x$ cm.; the distance of A from BC is 6 cm.; find the distance of D from AB.

2. In Fig. 177, AB$=$BP$=4''$, BC$=$PQ$=3''$, AC$=$BQ$=5''$; calculate the area common to $\triangle$s ABC, BPQ.

3. In $\triangle$ABC, AB$=$AC; P is any point on BC; Q, R are the mid-points of BP, PC; QX, RY are drawn perpendicular to BC and cut AB, AC at X, Y; prove BX$=$AY.

4. ABC is an equilateral triangle; BC is bisected at D and produced to E so that CE$=$CD, prove AE$^2=7$EC^2.

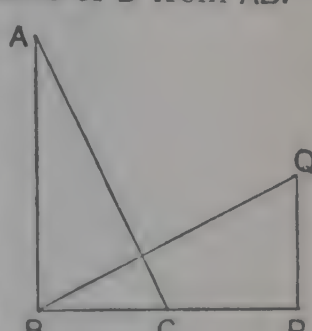


FIG. 177.

33.

1. In Fig. 178, the triangle ABC is inscribed in a rectangle: find its area and the distance of A from the mid-point of BC.

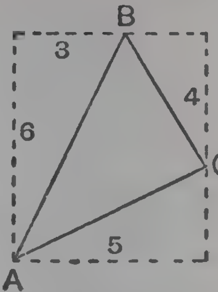


FIG. 178.

2. A, B are fixed points; X is a variable point such that $\angle AXB$ is obtuse; the perpendicular bisectors of AX, BX cut AB at Y, Z; prove that the perimeter of $\triangle XYZ$ is constant.

3. ABC is a $\triangle$; a line XY parallel to BC cuts AB, AC at X, Y, and is produced to Z so that $XZ=BC$; prove $\triangle BXY=\triangle AYZ$.

4. The sides of a triangle are 8 cm., 9 cm., 12 cm. Is it obtuse-angled?

34.

1. ABC is a triangle of area 24 sq. cm.; $AB=8$ cm., $AC=9$ cm.; D is a point on BC such that $BD=\frac{1}{3}BC$; find the distance of D from AB.

2. O is a point inside $\triangle ABC$ such that $OA=AC$; prove that $BA>AC$.

3. ABCD is a quadrilateral; AB is parallel to CD; BP, CP are drawn parallel to AC, AD to meet at P; prove $\triangle PDC=\triangle ABD$.

4. The length, breadth, and height of a room are each 10 feet; CAE, DBF are two vertical lines bisecting opposite walls, C, D being on the ceiling and E, F on the floor; $CA=x$ feet, $DB=4$ feet. Find in terms of x the shortest path from A to B—(i) along these two walls and the ceiling; (ii) along these two walls and one other wall. What is the condition that route (ii) is shorter than route (i)?

35.

1. In Fig. 179, $AB=9''$, $BC=8''$, $CD=7''$; if $AP=PD$, calculate BP.

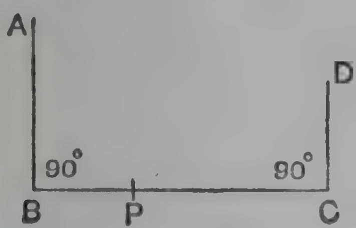


FIG. 179.

2. ABC is a $\triangle$; AP is the perpendicular from A to the bisector of $\angle ABC$; PQ is drawn parallel to BC to cut AB at Q; prove $AQ=QB=PQ$.

3. ABP, ABQ are equivalent triangles on opposite sides of AB; PR is drawn parallel to BQ to meet AB at R; prove QR is parallel to PB.

4. In $\triangle ABC$, $\angle BAC = 90°$; H, K are the mid-points of AB, AC; prove that $BH^2 + HK^2 + KC^2 = \frac{1}{2}BC^2$.

36.

1. The angles at the corners of Fig. 180 are all right angles. Construct a line parallel to AB to bisect the given figure. (The fact in Ex. 30, No. 63, may be useful.)

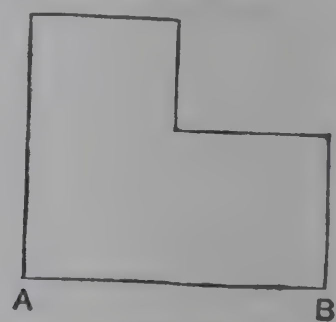


FIG. 180.

2. In $\triangle ABC$, $\angle BAC = 90°$; P, Q are the centres of the two squares which can be described on BC; prove that the distances of P, Q from AB are $\frac{1}{2}(AB \pm AC)$.

3. ABCD is a parallelogram; any line parallel to BA cuts BC, AC, AD at X, Y, Z; prove $\triangle AXY = \triangle DYZ$.

4. In $\triangle ABC$, $\angle ACB = 90°$; AD is a median; prove that $AB^2 = AD^2 + 3BD^2$.

SECTION IV.

CIRCLES.

THEOREM 31.

(1) The straight line which joins the centre of a circle to the middle point of a chord (which is not a diameter) is perpendicular to the chord.

(2) The line drawn from the centre of a circle perpendicular to a chord bisects the chord.

(1) *Given* a circle, centre O, and a chord AB, whose mid-point is N.

To prove $\angle ONA$ is a right angle.

Join OA, OB.

In the $\triangle$s ONA, ONB,

OA = OB, radii.

AN = BN, given.

ON is common.

$\therefore \triangle ONA \equiv \triangle ONB. \quad \therefore \angle ONA = \angle ONB.$

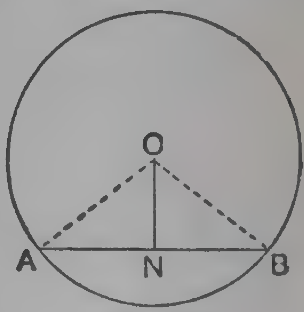


FIG. 181.

But these are adjacent angles, $\therefore$ each is a right angle.

(2) *Given* that ON is the perpendicular from the centre O of a circle to a chord AB.

To prove that N is the mid-point of AB.

In the *right-angled* triangles ONA, ONB.

OA = OB, radii.

ON is common.

$\therefore \triangle ONA \equiv \triangle ONB. \quad \therefore AN = NB.$ Q.E.D.

Corollary. The perpendicular bisector of a chord of a circle passes through the centre of the circle.

CONSTRUCTION 13.

Construct the centre of a circle, an arc of which is given.

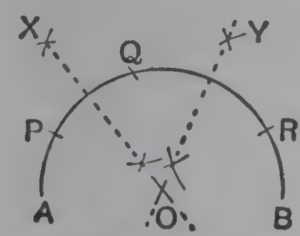


FIG. 182.

Given an arc AB of a circle.

To construct the centre of the circle.

Take three points P, Q, R on the arc.

Construct the perpendicular bisectors OX, OY of PQ, QR, intersecting at O.

Then O is the required centre.

Proof. The perpendicular bisector of a chord of a circle passes through the centre of the circle.

∴ the centre of the circle lies on OX and on OY.

∴ the centre is at O. Q.E.F.

CONSTRUCTION 14.

Construct a circle to pass through three given points, which do not lie on a straight line.

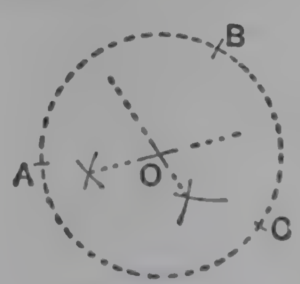


FIG. 183.

Given three points A, B, C.

To construct a circle to pass through A, B, C.

Construct the perpendicular bisectors OX, OY of AB, BC, intersecting at O.

With O as centre and OA as radius, describe a circle.
This is the required circle.

Proof. Since O lies on the perp. bisector of AB,

$$OA = OB.$$

Since O lies on the perp. bisector of BC,

$$OB = OC.$$

$$\therefore\ OA = OB = OC.$$

$\therefore$ the circle, centre O, radius OA, passes through B, C.

Q.E.F.

For examples on these constructions, see Exercise XLVI., No. 1-7.

Definition.

If ABC is any triangle, the circle which passes through A, B, C is called the **circum-circle** of the triangle, and the centre O of this circle is called the **circum-centre**. The radius of the circle is called the **circum-radius**, and the process of constructing the circle is described as **circumscribing a circle to a given triangle.**

From Construction 14 we see that if the three given points are not in the same straight line the perpendicular bisectors cannot be parallel, and therefore there is always one and only one position for the centre of the circle passing through them, and therefore one and only one such circle can be drawn. This result is expressed in the following theorem.

THEOREM 32.

There is one circle and only one circle which passes through three given points not in the same straight line.

Corollary. Two circles cannot intersect in more than two points.

THEOREM 33.

In equal circles or in the same circle:

(1) Equal chords are equidistant from the centres.

(2) Chords which are equidistant from the centres are equal.

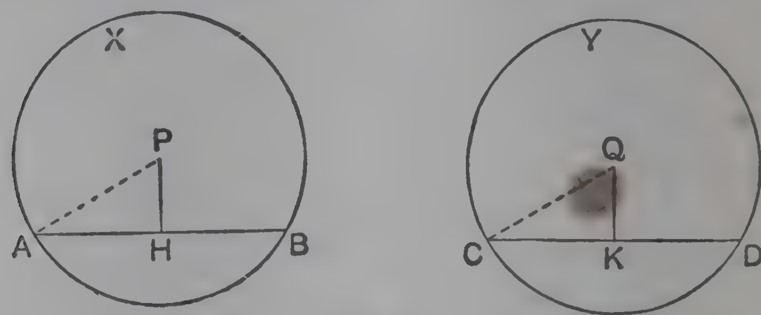


FIG. 184.

(1) *Given* two equal circles ABX, CDY, centres P, Q, and two equal chords AB, CD.

To prove that the perpendiculars PH, QK from P, Q to AB, CD are equal.

Join PA, QC.

Since PH, QK are the perpendiculars from the centres to the chords AB, CD, H and K are the mid-points of AB and CD.

$\therefore$ AH $= \frac{1}{2}$AB and CK $= \frac{1}{2}$CD.

But AB = CD, given.

$\therefore$ AH = CK.

$\therefore$ in the *right-angled* triangles PAH, QCK, the hypotenuse PA = the hypotenuse QC, radii of equal circles.

AH = CK, proved.

$\therefore$ $\triangle$PAH $\equiv$ $\triangle$QCK.

$\therefore$ PH = QK. Q.E.D.

(2) *Given* that the perpendiculars PH, QK from P, Q to the chords AB, CD are equal.

To prove that AB = CD.

In the *right-angled* triangles PAH, QCK, the hypotenuse PA = the hypotenuse QC, radii of equal circles.

PH = QK, given.

$\therefore$ $\triangle$PAH $\equiv$ $\triangle$QCK (rt. angle, hyp., side).

$\therefore$ AH = CK.

But the perpendiculars PH, QK bisect AB, CD.

$\therefore$ AB = 2AH and CD = 2CK.

$\therefore$ AB = CD. Q.E.D.

The proof is unaltered if the chords are in the same circle.

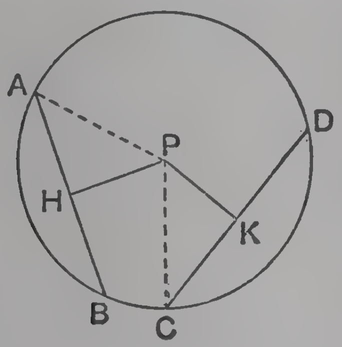


FIG. 185.

EXERCISE XXXVIII.

1. AB is a chord of a circle of radius 10 cm.; AB=8 cm.; find the distance of the centre of the circle from AB.

2. A chord of length 10 cm. is at a distance of 12 cm. from the centre of the circle; find the radius.

3. A chord of a circle of radius 7 cm. is at a distance of 4 cm. from the centre; find its length.

4. ABC is a △ inscribed in a circle; AB=AC=13″, BC=10″; calculate the radius of the circle.

5. In a circle of radius 5 cm., there are two parallel chords of lengths 4 cm., 6 cm.; find the distance between them.

6. Two parallel chords AB, CD of a circle are 3″ apart; AB=4″, CD=10″; calculate the radius of the circle.

7. An equilateral triangle, each side of which is 6 cm., is inscribed in a circle; find its radius.

8. The perpendicular bisector of a chord AB cuts AB at C and the circle at D; AB=6″, CD=1″; calculate the radius of the circle.

9. ABC is a straight line, such that AB=1″, BC=4″; PBQ is the chord of the circle on AC as diameter, perpendicular to AC; find PQ.

10. P is a point on the diameter AB of a circle; AP=2″, PB=8″; find the length of the shortest chord which passes through P.

11. Two concentric circles are of radii 3″, 5″; a line PQRS cuts one at P, S and the other at Q, R; if QR=2″, find PQ.

12. A crescent is formed of two circular arcs of equal radius (see Fig. 186); the perpendicular bisector of AB cuts the crescent at C, D; if CD=3 cm., AB=10 cm., find the radii.

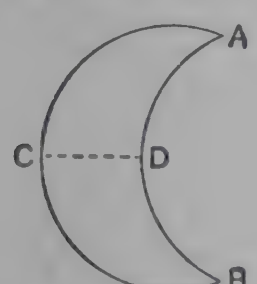


FIG. 186.

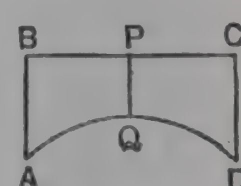


FIG. 187.

13. In Fig. 187, ABCD is the section of a lens; AB=CD=x; BP=PC=y; PQ=z; AB, QP, DC are perpendicular to BC; calculate in terms of x, y, z the radius of the circular arc AQD.

14. AB is a chord of a circle, centre O ; T is any point equidistant from A and B ; prove OT bisects ∠ATB.

15. Two circles, centres A, B, intersect at X, Y ; prove that AB bisects XY at right angles.

16. Two circles, centres A, B, intersect at C, D ; PCQ is a line parallel to AB cutting the circles at P, Q ; prove PQ=2AB.

17. Two circles, centres A, B, intersect at X, Y ; PQ is a chord of one circle parallel to XY ; prove AB bisects PQ.

18. A line PQRS cuts two concentric circles at P, S and Q, R ; prove PQ=RS.

19. ABC is a triangle inscribed in a circle ; if ∠BAC=90°, prove that the mid-point of BC is the centre of the circle.

20. In Fig. 188, if PQ is parallel to RS, prove PQ=RS.

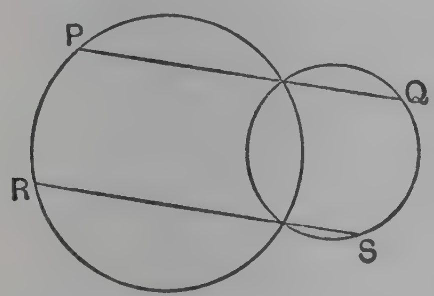


FIG. 188.

21. APB, CPD are intersecting chords of a circle, centre O ; if OP bisects ∠APC, prove AB=CD.

22. The diagonals of the quadrilateral ABCD meet at O ; circles are drawn through A, O, B ; B, O, C ; C, O, D ; D, O, A ; prove that their four centres are the corners of a parallelogram.

23. In Fig. 189, A, C, B are the centres of three unequal circles ; if AC=CB, prove PQ=RS.

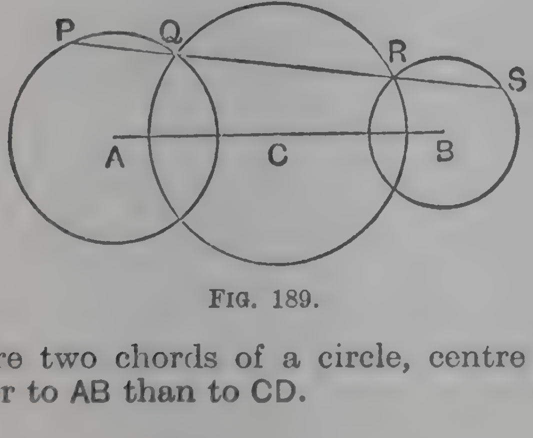


FIG. 189.

24. AB, CD are two chords of a circle, centre O ; if AB>CD, prove O is nearer to AB than to CD.

25. Two circles, centres A, B, intersect at C, D; PCQ is a line cutting the circles at P, Q; prove PQ is greatest when it is parallel to AB.

26. P is any point on a diameter AB of a circle; QPR is a chord such that $\angle APQ = 45°$; prove that $AB^2 = 2PQ^2 + 2PR^2$.

Definition.

The area bounded by two radii of a circle and the arc they cut off is called a **sector** of the circle.

The area bounded by a chord of a circle and the arc it cuts off is called a **segment** of the circle. A segment greater than a semicircle is called a **major** segment, if less a **minor** segment.

Any number of points are said to be **concyclic**, if a circle can be drawn to pass through all of them.

If the vertices of a quadrilateral lie on a circle, it is called a **cyclic quadrilateral.**

If ADC is any arc of a circle (see Fig. 193) and if B is any point on the remaining part of the circumference, the angle ABC is said to stand on the arc ADC.

N.B.—The arc on which an angle stands is the part of the circumference **intercepted between the arms of the angle.**

The following example is suggested for oral work:

Draw on the blackboard a large circle and mark (say) eight points A, B, C, D, E, F, G, H on its circumference.

Join every pair of points by a straight line, and consider such questions as:

(i) On what arcs do the angles BCE, HAB, etc. stand?

(ii) What angles stand on the arc EGA?

(iii) What angles stand on the same arc as the angle HAD?

(iv) What angles stand on the chord AC?

(v) Name two angles standing on the same chord, but not on the same arc.

THEOREM 34.

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

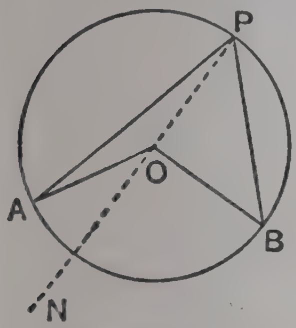


FIG. 190 (1).

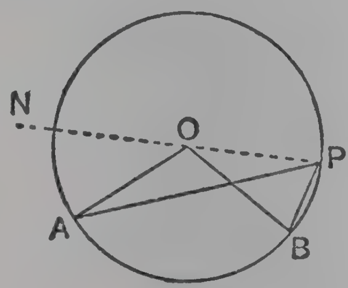


FIG. 190 (2).

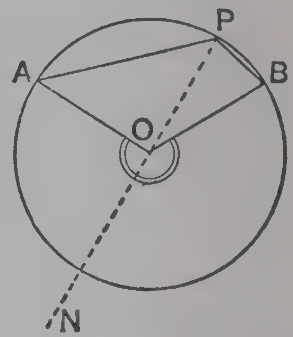


FIG. 190 (3).

Given AB is an arc of a circle, centre O; P is any point on the remaining part of the circumference.

To prove $\angle AOB = 2\angle APB$.

Join PO, and produce it to any point N.

Since $OA = OP$, radii, $\angle OAP = \angle OPA$.

But ext. $\angle NOA =$ int. $\angle OAP +$ int. $\angle OPA$.

$\therefore$ $\angle NOA = 2\angle OPA$.

Similarly $\angle NOB = 2\angle OPB$.

$\therefore$ adding in Fig. 190 (1) and subtracting in Fig. 190 (2), we have $\angle AOB = 2\angle APB$. Q.E.D.

Fig. 190 (3) shows the case where the angle AOB is reflex, *i.e.* greater than 180°; the proof for Fig. 190 (3) is the same as for Fig. 190 (1).

THEOREM 35.

(1) Angles in the same segment of a circle are equal.

(2) The angle in a semicircle is a right angle.

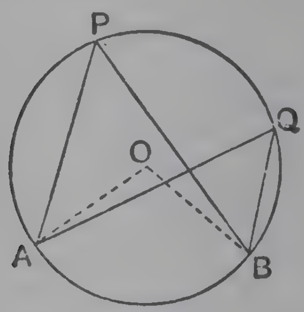


FIG. 191 (1).

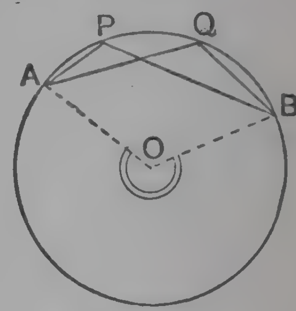


FIG. 191 (2).

(1) *Given* two angles APB, AQB in the same segment of a circle.

To prove $\angle APB = \angle AQB$.

Let O be the centre. Join OA, OB.

Then $\angle AOB = 2\angle APB$. $\angle$ at centre = twice $\angle$ at ○ce.

and $\angle AOB = 2\angle AQB$.

$\therefore$ $\angle APB = \angle AQB$. Q.E.D.

(2) *Given* AB a diameter of a circle, centre O, and P a point on the circumference.

To prove $\angle APB = 90°$.

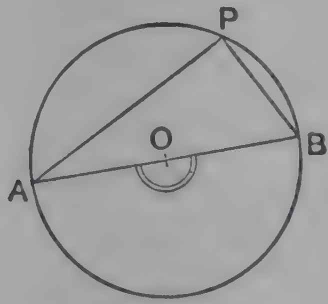


FIG. 192.

$\angle AOB = 2\angle APB$. $\angle$ at centre = twice $\angle$ at ○ce.

But $\angle AOB = 180°$, since AOB is a straight line ;

$\therefore$ $\angle APB = 90°$. Q.E.D.

THEOREM 36.

(1) The opposite angles of a cyclic quadrilateral are supplementary.

(2) If a side of a cyclic quadrilateral is produced, the exterior angle is equal to the interior opposite angle.

(1) *Given* ABCD is a cyclic quadrilateral.

To prove $\angle ABC + \angle ADC = 180°$.

Let O be the centre of the circle. Join OA, OC.

Let the arc ADC subtend angle $x°$ at the centre, and let the arc ABC subtend angle $y°$ at the centre.

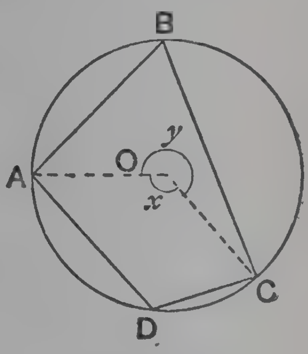


FIG. 193.

$$\therefore\ x° + y° = 360°.$$

Now $x° = 2\angle ABC$. $\angle$ at centre = twice $\angle$ at ○ce.

and $y° = 2\angle ADC$.

$$\therefore\ 2\angle ABC + 2\angle ADC = 360°.$$

$$\therefore\ \angle ABC + \angle ADC = 180°.$$ Q.E.D.

(2) *Given* the side AD of the cyclic quadrilateral ABCD is produced to P.

To prove $\angle PDC = \angle ABC$.

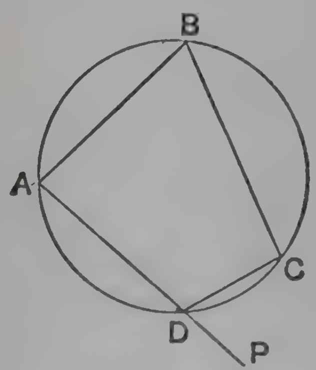


FIG. 194.

Now $\angle ADC + \angle PDC = 180°$, adj. angles, ADP a st. line.

and $\angle ADC + \angle ABC = 180°$, opp. $\angle$s cyclic quad.

$$\therefore\ \angle ADC + \angle PDC = \angle ADC + \angle ABC.$$

$$\therefore\ \angle PDC = \angle ABC.$$ Q.E.D.

EXERCISE XXXIX.

1. ABC is a $\triangle$ inscribed in a circle, centre O; $\angle AOC=130°$, $\angle BOC=150°$, find $\angle ACB$.

2. AB, CD are perpendicular chords of a circle; $\angle BAC=35°$, find $\angle ABD$.

3. ABCD is a quadrilateral such that AB=AC=AD; if $\angle BAD =140°$, find $\angle BCD$.

4. ABCD is a quadrilateral inscribed in a circle; AB is a diameter; $\angle ADC=127°$; find $\angle BAC$.

5. Two chords AB, CD when produced meet at O; $\angle OAD=31°$; $\angle AOC=42°$; find $\angle OBC$.

6. Two circles APRB, AQSB intersect at A, B; PAQ, RBS are straight lines; if $\angle QPR=80°$, $\angle PRS=70°$, find $\angle PQS$, $\angle QSR$.

7. P, Q, R are points of a circle, centre O; $\angle POQ=54°$, $\angle OQR =36°$; P, R are on opposite sides of OQ; find $\angle QPR$ and $\angle PQR$.

8. The diagonals of the cyclic quadrilateral ABCD meet at O; $\angle BAC=42°$, $\angle BOC=114°$, $\angle ADB=33°$; find $\angle BCD$.

9. ABCD is a cyclic quadrilateral, EABF is a straight line; $\angle EAD=82°$, $\angle FBC=74°$, $\angle BDC=50°$; find angle between AC, BD.

10. Two chords AB, DC of a circle, centre O, are produced to meet at E; $\angle AOB=100°$, $\angle EBC=72°$, $\angle ECB=84°$; find $\angle COD$.

11. (i) In Fig. 195, if $y=32°$, $z=40°$, find x.
(ii) If $y+z=90°$, prove $x=45°$.

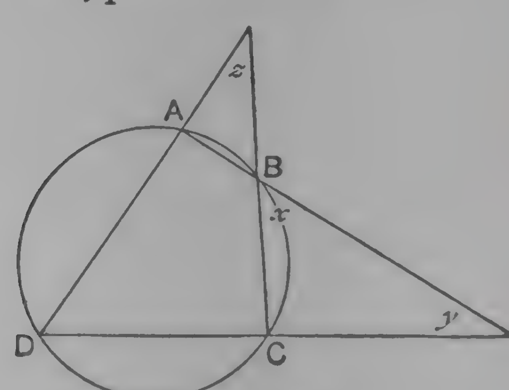


FIG. 195.

12. D is a point on the base BC of $\triangle ABC$; H, K are the centres of the circles ADB, ADC; if $\angle AHD=70°$, $\angle AKD=80°$, find $\angle BAC$.

13. In Fig. 195, if AC cuts BD at O, if $y=20°$, $z=40°$, $\angle BOC =100°$, prove $\angle BAC=2\angle BCA$.

14. AB, XY are parallel chords of a circle; AY cuts BX at O; prove OX=OY.

15. Two circles BAPR, BASQ cut at A, B; PAQ, RAS are straight lines; prove $\angle$PBR=$\angle$QBS.

16. AB is a chord of a circle, centre O; P is any point on the minor arc AB; prove $\angle$AOB+2$\angle$APB=360°.

17. ABCD is a cyclic quadrilateral; if AC bisects the angles at A and C, prove $\angle$ABC=90°.

18. Two lines OAB, OCD cut a circle at A, B and C, D; prove $\angle$OAD=$\angle$OCB.

19. AB is a diameter of a circle APQRB; prove $\angle$APQ+$\angle$QRB =270°.

20. ABCDEF is a hexagon inscribed in a circle; prove that $\angle$FAB+$\angle$BCD+$\angle$DEF=360°.

21. Two circles ABPR, ABQS cut at A, B; PBQ, RAS are straight lines; prove PR is parallel to QS.

22. A, B, C are three points on a circle, centre O; prove that $\angle$BAC=$\angle$OBA$\pm$$\angle$OCA.

23. A, B, C, P are four points on a circle; prove that a triangle whose sides are parallel to PA, PB, PC is equiangular to $\triangle$ABC.

24. AP, AQ are diameters of the circles APB, AQB; prove that PBQ is a straight line.

25. OA is a radius of a circle, centre O; AP is any chord; prove that the circle on OA as diameter bisects AP.

26. Two chords AOB, COD of a circle intersect at O; if AO=AC, prove DO=BD.

27. APC is an arc, less than a semicircle, of a circle, centre O; AQOC is another circular arc; prove $\angle$APC=$\angle$PAQ+$\angle$PCQ.

28. If in Fig. 196, O is the centre of the circle, prove that $x+y=z$.

29. ABC is a $\triangle$ inscribed in a circle, centre O; D is the mid-point of BC; prove $\angle$BOD=$\angle$BAC.

30. OA, OB, OC are three equal lines; if $\angle$AOB=90°, prove $\angle$ACB=45° or 135°.

31. Two lines OAB, OCD cut a circle at A, B and C, D; if OB=BD, prove OC=CA.

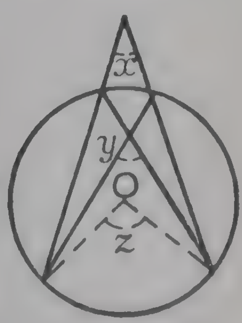


FIG. 196.

32. In Fig. 197, AB=AC and C is the centre of the circle; prove that DE is parallel to the line bisecting $\angle$ABC.

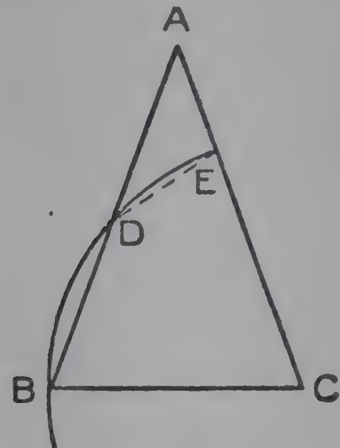


FIG. 197.

33. ABCD is a rectangle; any circle through A cuts AB, AC, AD at X, Y, Z; prove that ABD, XYZ are equiangular triangles.

34. In Fig. 198, O is the centre of the circle; prove $\angle$AOC + $\angle$BOD=2$\angle$AEC.

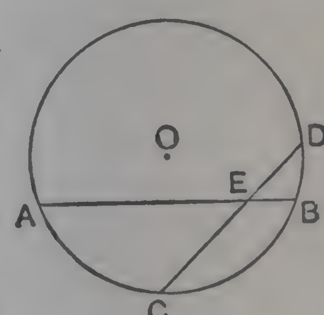


FIG. 198.

35. ABCD is a cyclic quadrilateral; AD, BC are produced to meet at E; AB, DC are produced to meet at F; the circles EDC, FBC cut at X; prove EXF is a straight line.

36. AB, CD are perpendicular chords of a circle, centre O; prove $\angle$DAB=$\angle$OAC.

37. In $\triangle$ABC, AB=AC; ABD is an equilateral triangle; prove that $\angle$BCD=30° or 150°.

38. ABC is a $\triangle$; the bisectors of $\angle$s ABC, ACB intersect at I and meet AC, AB at P, Q; if A, Q, I, P are concyclic, prove $\angle$BAC=60°.

39. Two lines EBA, ECD cut a circle ABCD at B, A and C, D; O is the centre; prove $\angle$AOD $-$ $\angle$BOC=2$\angle$BEC.

40. ACB, ADB are two arcs on the same side of AB; a straight line ACD cuts them at C, D; if the centre of the circle ADB lies on the arc ACB, prove CB=CD.

41. ABCD is a quadrilateral inscribed in a circle; BA, CD when produced meet at E; O is the centre of the circle EAC; prove that BD is perpendicular to OE.

42. ABC is a $\triangle$ inscribed in a circle; AOX, BOY, COZ are three chords intersecting at a point O inside $\triangle ABC$; prove $\angle YXZ = \angle BOC - \angle BAC$.

43. D is any point on the side AB of $\triangle ABC$; points E, F are taken on AC, BC so that $\angle EDA = 60° = \angle FDB$; a circle is drawn through D, E, F and cuts AB again at G; prove $\triangle EFG$ is equilateral.

44. ABC is a $\triangle$; a line PQR cuts BC produced, CA, AB at P, Q, R; if B, C, Q, R are concyclic, prove the bisectors of $\angle$s BPR, BAC are at right angles.

45. ABC is a $\triangle$; the bisectors of $\angle$s ABC, ACB meet at I; the circle BIC cuts AB, AC again at P, Q; prove AB=AQ and AC=AP.

46. ABC is a $\triangle$; the bisectors of $\angle$s ABC, ACB intersect at I, and cut AC, AB at Y, Z; the circles BIZ, CIY meet again at X; prove $\angle YXZ + \angle BIC = 180°$.

47. ABC is a triangle inscribed in a circle; AB=AC; BC is produced to D; AD cuts the circle at E; prove $\angle ACE = \angle ADB$.

48. AOB, COD are perpendicular chords of a circle ACBD; prove that the perpendicular from O to AD bisects, when produced, BC.

49. Two given circles ABP, ABQ intersect at A, B; a variable line PAQ meets them at P, Q; prove $\angle PBQ$ is of constant size.

50. ABC is a given $\triangle$; P is a variable point on a given circle which passes through B, C; if P, A are on the same side of BC, prove $\angle PBA - \angle PCA$ is constant.

51. In Fig. 199, the circles are given; prove $\angle PRQ$ is of constant

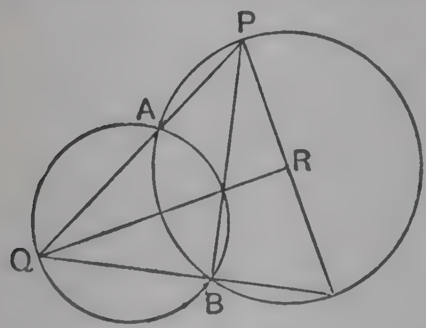


FIG. 199.

Any three points lie either on a straight line or a circle. In general, if any four points are taken, it is impossible to draw either a straight line or a circle to pass through all of them. The next theorems give two tests for determining whether four points are concyclic. These tests are the converses of Theorems 35, 36.

THEOREM 37.

If the line joining two points subtends equal angles at two other points on the same side of it, then the four points lie on a circle.

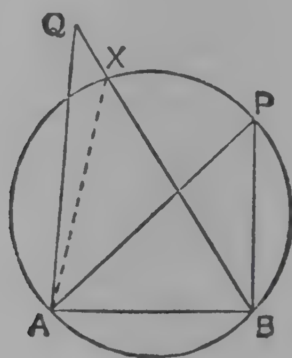


FIG. 200 (1).

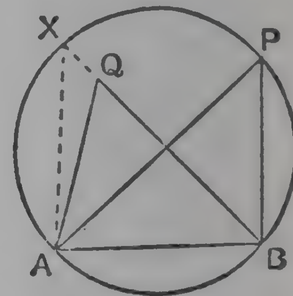


FIG. 200 (2).

Given that $\angle APB = \angle AQB$, where P, Q lie on the same side of AB.

To prove that A, P, Q, B lie on a circle.

One of the angles ABP, ABQ must be the greater, suppose it is $\angle ABP$, so that BQ lies in the angle ABP.

Draw the circle through A, B, P and suppose, if possible, that it does not pass through Q.

Since BQ lies in the angle ABP, the circle must cut BQ or BQ produced, at X say.

Then $\angle AXB = \angle APB$, same segment.

But $\angle AQB = \angle APB$, given.

$\therefore$ $\angle AXB = \angle AQB$.

But one of these is the exterior angle and the other is the interior opposite angle of the triangle AQX.

$\therefore$ they cannot be equal.

$\therefore$ the circle through A, B, P must pass through Q.

Q.E.D.

THEOREM 38.

If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

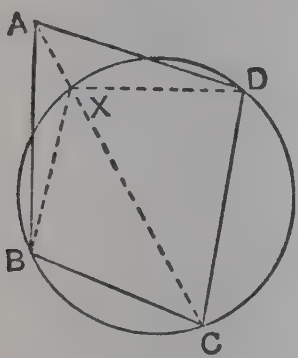


FIG. 201 (1).

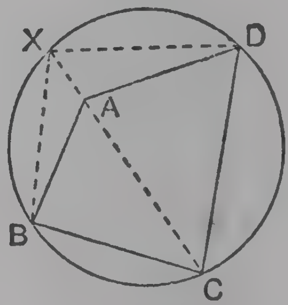


FIG. 201 (2).

Given that in the quadrilateral ABCD, $\angle ABC + \angle ADC = 180°$.

To prove that A, B, C, D lie on a circle.

Draw the circle through B, C, D and suppose, if possible, that it does not pass through A.

Since CA lies in the angle BCD, the circle must cut either CA or CA produced, at X say.

Then $\angle ABC + \angle ADC = 180°$, given.

But $\angle XBC + \angle XDC = 180°$, opp. $\angle$s cyclic quad.

$\therefore \angle ABC + \angle ADC = \angle XBC + \angle XDC$.

But one of these is only a part of the other, therefore they cannot be equal.

$\therefore$ the circle through B, C, D must pass through A.

Q.E.D.

THEOREM 39.

The circle described on the hypotenuse of a right-angled triangle as diameter passes through the opposite vertex.

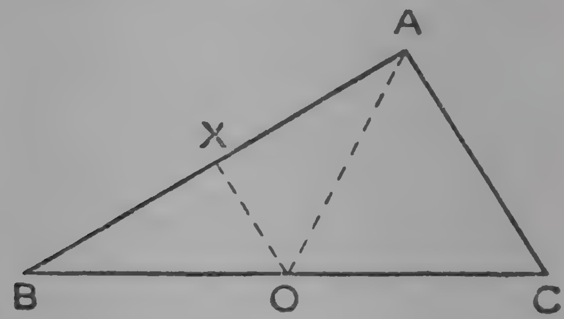


FIG. 202.

Given a triangle ABC, right-angled at A.

To prove the circle on BC as diameter passes through A.

Bisect BC at O; through O draw OX parallel to CA to cut AB at X; join OA.

Since BO = OC and OX is parallel to CA.

$\therefore$ OX bisects BA.

Since $\angle$BAC = 90° and OX is parallel to CA.

$\therefore$ OX is perpendicular to BA.

$\therefore$ OX is the perpendicular bisector of BA.

$\therefore$ OA = OB.

But OB = OC, $\therefore$ OA = OB = OC.

$\therefore$ the circle on BC as diameter passes through A.

Q.E.D.

Note.—This theorem is really a special case of Theorem 37. making use of Theorem 35 (2).

EXERCISE XL.

1. ABCD is a parallelogram; if $\angle ABC = 60°$, prove that the centre of the circle ABD lies on the circle CBD.

2. BE, CF are altitudes of $\triangle ABC$; prove that $\angle AEF = \angle ABC$.

3. The altitudes AD, BE of $\triangle ABC$ intersect at H; prove that $\angle DHC = \angle DEC$.

4. ABCD is a parallelogram: any circle through A, D cuts AB, DC at P, Q; prove that B, C, Q, P are concyclic.

5. The circle BCGF lies inside the circle ADHE; OABCD and OEFGH are two lines cutting them; if A, B, F, E are concyclic, prove that C, D, H, G are concyclic.

6. In Fig. 203, BQP and BAC are equiangular isosceles triangles; prove that QA is parallel to BC.

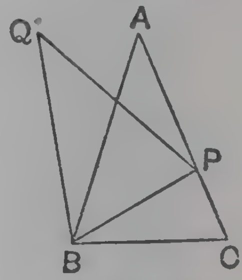


FIG. 203.

7. ABCD is a rectangle; the line through C perpendicular to AC cuts AB, AD produced at P, Q; prove that P, D, B, Q are concyclic.

8. O is a fixed point inside a given $\triangle ABC$; X is a variable point on BC; the circles BXO, CXO cuts AB, AC at Z, Y; prove that (1) O, Y, A, Z are concyclic, (2) the angles of $\triangle XYZ$ are of constant size.

9. AB, CD are two intersecting chords of a circle; AP, CQ are the perpendiculars from A, C to CD, AB; prove that PQ is parallel to BD.

10. Prove that the quadrilateral formed by the external bisectors of the angles of any quadrilateral is cyclic.

11. If any five circles are drawn intersecting as in Fig. 204, prove that a sixth circle can be drawn to pass through P, Q, R, S.

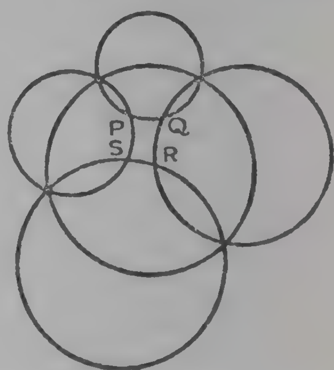


FIG. 204.

12. AOB, COD are two perpendicular chords of a circle; DE is any other chord; AF is the perpendicular from A to DE; prove that OF is parallel to BE.

13. BE, CF are altitudes of $\triangle ABC$; X is the mid-point of BC; prove that $XE = XF$.

14. BE, CF are altitudes of $\triangle ABC$; X is the mid-point of BC; prove that $\angle FXE = 180° - 2\angle BAC$.

15. Two circles APRB, ASQB intersect at A, B; PAQ and RAS are straight lines; RP and QS are produced to meet at O; prove that O, P, B, Q are concyclic.

16. AOB, COD are two perpendicular diameters of a circle; two chords CP, CQ cut AB at H, K; prove that H, K, Q, P are concyclic.

17. X, Y are the centres of the circles ABP, ABQ; PAQ is a straight line; PX and QY are produced to meet at R; prove that X, Y, B, R are concyclic.

18. BE, CF are altitudes of $\triangle ABC$; Z is the mid-point of AB; prove that $\angle ZEF = \angle ABC \sim \angle BAC$.

19. ABCD is a parallelogram; O is a point inside ABCD such that $\angle AOB + \angle COD = 180°$; prove that $\angle OBC = \angle ODC$.

Equal Arcs of the same or equal Circles.

Given a circle, centre O, and an arc AB of the circle, what measurements must be made in order to make an accurate copy of the arc AB?

The size of the circle is fixed uniquely if we measure its radius. Further, since the circle is symmetrical about any diameter the size of the arc AB is fixed by measuring $\angle AOB$.

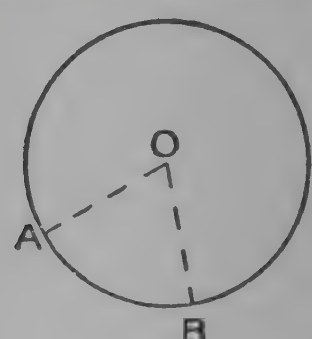


FIG. 205.

Consequently, if a number of circular arcs are drawn which agree with each other as regards both (i) the length of the radius, (ii) the angle subtended at the centre of the circle, then they will agree completely in size and shape, or in other words will be congruent, and in particular will be of equal length. (If any two circular arcs are of equal length, must they be congruent?)

This result may be expressed as follows:

THEOREM 40.

(i) In equal circles or in the same circle equal angles at the centre stand on equal arcs.

(ii) In equal circles or in the same circle equal arcs subtend equal angles at the centre.

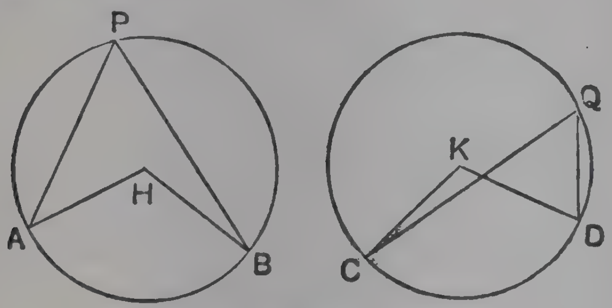


Fig. 206.

Corollary. In equal circles or in the same circle equal angles at the circumference stand on equal arcs, and conversely equal arcs subtend equal angles at the circumference.

This follows at once from the fact that the angle at the centre is double the angle at the circumference, standing on the same arc.

THEOREM 41.

(i) In equal circles or in the same circle equal chords cut off equal arcs.

(ii) In equal circles or in the same circle the chords of equal arcs are equal.

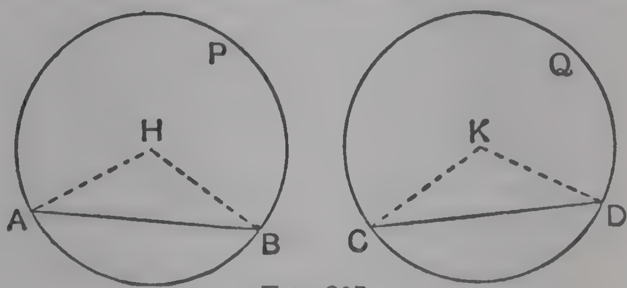


Fig. 207.

(i) In Fig. 207, if chord AB = chord CD, then $\triangle AHB \equiv \triangle CKD$, having three sides equal.

$\therefore$ $\angle AHB = \angle CKD$, so that arc AB = arc CD.

(ii) In Fig. 207, if arc AB = arc CD, then $\angle AHB = CKD$.

$\therefore$ $\triangle AHB \equiv \triangle CKD$, two sides and included angle;

$\therefore$ chord AB = chord CD. Q.E.D.

EXERCISE XLI.

1. ABCD is a square and AEF is an equilateral triangle inscribed in the same circle; calculate the angles of $\triangle$ECD.

2. AB is a side of a regular hexagon and AC of a regular octagon inscribed in the same circle; calculate the angles of $\triangle$ABC.

3. ABCD is a quadrilateral inscribed in a circle; AC cuts BD at O: DA, CB when produced meet at E; AB, DC when produced meet at F; if $\angle$AEB$=55°$, $\angle$BFC$=35°$, $\angle$DOC$=85°$, prove arc BC=twice arc AB.

4. ABC is a triangle inscribed in a circle: a line through A meets BC produced at T; $\angle$BAT$=135°$, $\angle$ATB$=30°$; find the ratio of the arcs AB and AC, if $\angle$TAC$=\angle$ABC.

5. A, B are two points on the circle ABCD such that the minor arc AB is half the major arc AB; $\angle$DAB$=74°$; arc BC=arc CD; calculate $\angle$ABD and $\angle$BDC.

6. ABCD is a quadrilateral inscribed in a circle; $\angle$ADB$=25°$, $\angle$DBC$=65°$; prove arc AB+arc CD=arc BC+arc AD.

---

7. AB, DC are parallel chords of a circle; prove arc AD=arc BC.

8. ABCD is a cyclic quadrilateral; if AB=CD, prove $\angle$ABC $=\angle$BCD.

9. A circle AOBP passes through the centre O of a circle ABQ; prove that OP bisects $\angle$APB.

10. ABP, ABQ are two equal circles; PBQ is a straight line; prove AP=AQ.

11. AB, BC, CD are equal chords of a circle, centre O; prove that AC cuts BD at an angle equal to $\angle$AOB.

12. ABCD is a square and APQ an equilateral triangle inscribed in the same circle, P being between B and C; prove arc BP=$\frac{1}{2}$ arc PC.

13. On a clock-face, prove that the line joining 4 and 7 is perpendicular to the line joining 5 and 12.

14. X, Y are the mid-points of the arcs AB, AC of a circle; XY cuts AB, AC at H, K; prove AH=AK.

15. If AB and CD are equal arcs of a circle, prove that the chords BC, AD are either equal or parallel.

16. ABCD is a rectangle inscribed in a circle; DP is a chord equal to DC; prove PB=AD.

17. A hexagon is inscribed in a circle; if two pairs of opposite sides are parallel, prove that the third pair is also parallel.

18. ABCDEF is a hexagon inscribed in a circle; if $\angle ABC = \angle DEF$, prove AF is parallel to CD.

19. CD is a quadrant of the circle ACDB; AB is a diameter; AD cuts BC at P; prove AC=CP.

20. ABC is a $\triangle$ inscribed in a circle, centre O; P is any point on the side BC; prove that the circles OBP, OCP are equal.

21. In $\triangle ABC$, AB=AC; BC is produced to D; prove that the circles ABD, ACD are equal.

22. ABCD is a quadrilateral inscribed in a circle; CD is produced to F; the bisector of $\angle ABC$ cuts the circle at E; prove that DE bisects $\angle ADF$.

23. ABCD is a cyclic quadrilateral; BC and AD are produced to meet at E; a circle is drawn through A, C, E and cuts AB, CD again at P, Q; prove PE=EQ.

24. AB, AC are equal chords of a circle; BC is produced to D so that CD=CA; DA cuts the circle at E; prove that BE bisects $\angle ABC$.

25. ABC is an equilateral triangle inscribed in a circle; H, K are the mid-points of the arcs AB, AC; prove that HK is trisected by AB, AC.

26. AB, BC are two chords of a circle (AB>BC); the minor arc AB is folded over about the chord AB and cuts AC at D; prove BD=BC.

27. ABCD is a quadrilateral inscribed in a circle; X, Y, Z, W are the mid-points of the arcs AB, BC, CD, DA; prove that XZ is perpendicular to YW.

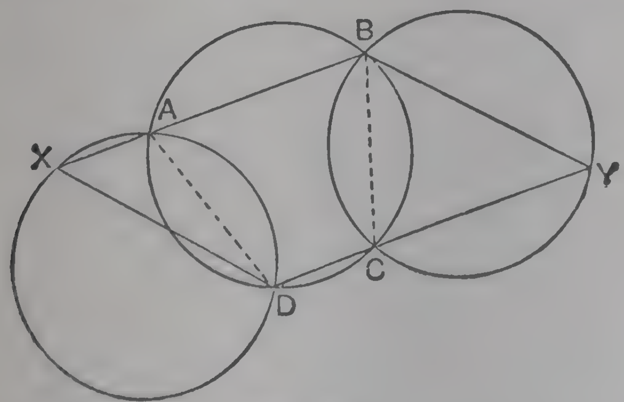


FIG. 208.

28. In Fig. 208, the circles are equal and AD=BC; prove XBYD is a parallelogram.

29. In $\triangle ABC$, $AB > AC$; the bisectors of $\angle$s ABC, ACB meet at I; the circle BIC cuts AB, AC, at P, Q; prove $PI = IC$ and $QI = IB$.

30. ABC is a triangle inscribed in a circle, centre O; PQ is the diameter perpendicular to BC, P and A being on the same side of BC; prove $\angle ABC \sim \angle ACB = \angle POA$.

31. ABC is an equilateral triangle inscribed in a circle; D, E are points on the arcs AB, BC such that $AD = BE$, prove $AD + DB = AE$.

32. Two fixed circles cut at A, B; P is a variable point on one; PA, PB when produced cut the other at QR; prove QR is of constant length.

The Tangent to a Circle.

Definitions.

If a straight line cuts a circle at two distinct points it is called a **secant**.

If a straight line has one point, and only one point, in common with a circle, however far it is produced, the straight line is called a **tangent** to the circle, and the common point is called the **point of contact**.

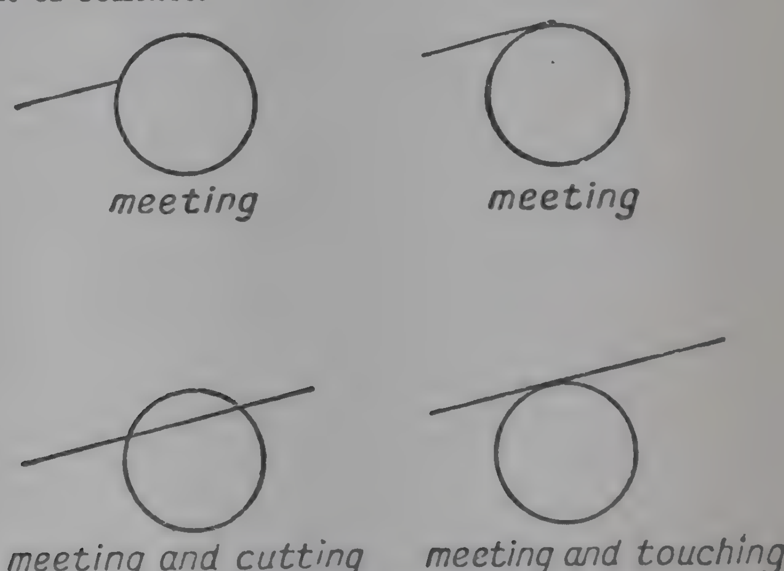


FIG. 209.

The words "touching" and "meeting" must not be confused. If a line meets a circle, it may when produced meet it at a second distinct point, and if it does so the line is a secant: on the other hand it may only have one point in common with the circle, however far either way the line is produced, and, if this is so, the line is a tangent.

THEOREM 42.

(1) The straight line drawn perpendicular to a radius of a circle at its extremity is a tangent to the circle.

(2) A tangent to a circle is perpendicular to the radius drawn through the point of contact.

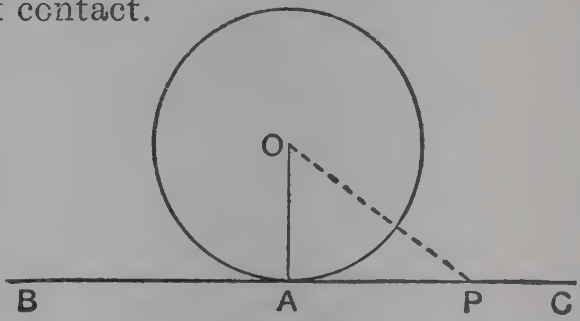


FIG. 210.

(1) *Given* that O is the centre and OA a radius of a circle, and that BAC is a line perpendicular to OA.

To prove that BAC touches the circle at A.

Let P be any point on BC; join OP.

$\angle OAP = 90°$ given, $\therefore$ each other angle of $\triangle OAP$ is less than $90°$.

$$\therefore \angle OPA < \angle OAP. \qquad \therefore OA < OP.$$

But OA is a radius, $\therefore$ P lies outside the circle.

Similarly every point on BC except A lies outside the circle.

$\therefore$ BC touches the circle at A.　　Q.E.D.

(2) *Given* that the line BAC touches the circle, centre O, at A.

To prove that $\angle OAC = 90°$.

If OA is not perpendicular to BC, draw OP perpendicular to BC.

Since $\angle OPA = 90°$, each other angle of $\triangle OAP$ is less than $90°$.

$$\therefore \angle OAP < \angle OPA. \qquad \therefore OP < OA.$$

But OA is a radius, $\therefore$ P lies inside the circle.

$\therefore$ AP if produced must cut the circle again, which is impossible since AP is a tangent.

$\therefore$ OA must be perpendicular to BC.　　Q.E.D.

Corollary 1. *At every point of a circle, one and only one tangent can be drawn to the circle.*

The line through the point perpendicular to the radius is a tangent; $\therefore$ there is one tangent. Further there cannot be more than one tangent, because it must be perpendicular to the radius through the point.

Corollary 2. *The perpendicular to a tangent at its point of contact passes through the centre of the circle.*

For the radius through the point of contact is this perpendicular.

THEOREM 43.

If a straight line touches a circle and, from the point of contact, a chord is drawn, the angles which the chord makes with the tangent are equal to the angles in the alternate segments of the circle.

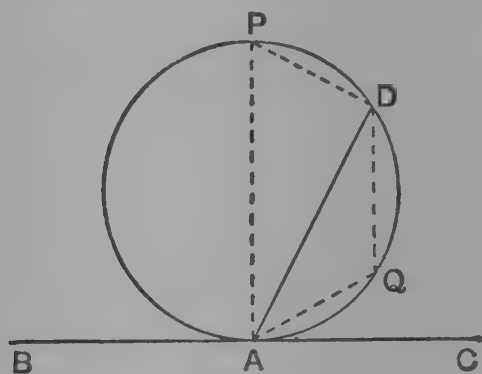


FIG. 211.

Given that the line BAC touches the circle at A and that AD is any chord.

To prove (i) $\angle DAC =$ angle in alternate segment APD.

(ii) $\angle DAB =$ angle in alternate segment AQD.

(i) Draw the diameter AP through A.

Since AP passes through the centre of the circle and since AC is a tangent $\qquad \angle PAC = 90°$.

Since AP is a diameter, $\qquad \angle ADP = 90°$.

But the sum of the angles of $\triangle APD$ is $180°$.

$$\therefore \angle PAD + \angle APD = 90°.$$

$$\therefore \angle PAD + \angle APD = \angle PAC = \angle PAD + \angle DAC.$$

$$\therefore \angle APD = \angle DAC.$$

$\therefore \angle DAC =$ angle in alternate segment APD.

(ii) Take any point Q on the minor arc AD.

APDQ is a cyclic quadrilateral, $\therefore \angle AQD + \angle APD = 180°$.

Also BAC is a straight line, $\therefore \angle BAD + \angle DAC = 180°$.

$$\therefore \angle BAD + \angle DAC = \angle AQD + \angle APD.$$

But $\angle DAC = \angle APD$, proved.

$$\therefore \angle BAD = \angle AQD.$$

$\therefore \angle BAD =$ angle in alternate segment AQD. Q.E.D.

EXERCISE XLII.

1. A line TBC cuts a circle ABC at B, C; TA is a tangent; if $\angle TAC=118°$, $\angle ATC=26°$, find $\angle ABC$.

2. ABC is a minor arc of a circle; the tangents at A, C meet at T; if $\angle ATC=54°$, find $\angle ABC$.

3. AOC, BOD are chords of a circle ABCD; the tangent at A meets DB produced at T; if $\angle ATD=24°$, $\angle COD=82°$, $\angle TBC=146°$, find $\angle BAC$. Find also the angle between BD and the tangent at C.

4. The sides BC, CA, AB of a $\triangle$ touch a circle at X, Y, Z; $\angle ABC=64°$, $\angle ACB=52°$; find $\angle XYZ$, $\angle XZY$.

5. Three of the angles of a quadrilateral circumscribing a circle are 70°, 84°, 96° in order; find the angles of the quadrilateral whose vertices are the points of contact.

6. TBP, TCQ are tangents to the circle ABC; $\angle PBA=146°$, $\angle QCA=128°$; find $\angle BAC$ and $\angle BTC$.

7. In $\triangle ABC$, $\angle ABC=50°$, $\angle ACB=70°$; a circle touches BC, AC produced, AB produced at X, Y, Z; find $\angle YXZ$.

---

8. A chord AB of a circle is produced to T; TC is a tangent from T to the circle; prove $\angle TBC=\angle ACT$.

9. Two circles APB, AQB intersect at A, B: AP, AQ are the tangents at A, prove $\angle ABP=\angle ABQ$.

10. DA is the tangent at A to the circle ABC; if DB is parallel to AC, prove $\angle ADB=\angle ABC$,

11. In $\triangle ABC$, AB=AC; D is the mid-point of BC; prove that the tangent at D to the circle ADC is perpendicular to AB.

12. BC, AD are parallel chords of the circle ABCD; the tangent at A cuts CB produced at P; PD cuts the circle at Q; prove $\angle PAQ=\angle BPQ$.

13. Two circles ACB, ADB intersect at A, B; CA, DB are tangents to circles ADB, ACB at A, B; prove AD is parallel to BC.

14. CA, CB are equal chords of a circle; the tangent ADE at A meets BC produced at D; prove $\angle BDE=3\angle CAD$.

15. The bisector of $\angle BAC$ meets BC at D; a circle is drawn touching BC at D and passing through A; if it cuts AB, AC at P, Q, prove $\angle PDB=\angle QDC$.

16. Two circles APB, AQB intersect at A, B; AQ, AP are the tangents at A; if PBQ is a straight line, prove $\angle PAQ=90°$.

17. ABCD is a quadrilateral inscribed in a circle; the tangents at A, C meet at T; prove $\angle ATC = \angle ABC \sim \angle ADC$.

18. Two circles intersect at A, B; the tangents at B meet the circles at P, Q; if $\angle PBQ$ is acute, prove $\angle PAQ = 2\angle PBQ$. What happens if $\angle PBQ$ is obtuse?

19. ABC is a $\triangle$ inscribed in a circle; the tangent at C meets AB produced at T; the bisector of $\angle ACB$ cuts AB at D; prove TC=TD.

20. AOB is a diameter of a circle, centre O; the tangent at B meets any chord AP at T; prove $\angle ATB = \angle OPB$.

21. In $\triangle ABC$, AB=AC; a circle is drawn to touch BC at B and to pass through A; if it cuts AC at D, prove BC=BD.

22. In $\triangle ABC$, $\angle BAC = 90°$; D is any point on BC; DP, DQ are tangents at D to the circles ABD, ACD; prove $\angle PDQ = 90°$.

23. AB is a diameter of a circle ABC; TC is the tangent from a point T on AB produced; TD is drawn perpendicular to TA and meets AC produced at D; prove TC=TD.

24. Two circles touch internally at A; the tangent at any point P on the inner cuts the outer at Q, R; AQ, AR cut the inner at H, K; prove $\triangle$s PQH, APK are equiangular.

25. PQ is a common tangent to two circles CDP, CDQ; prove that $\angle PCQ + \angle PDQ = 180°$.

26. Two chords AOB, COD of a circle cut at O; the tangents at A, C meet at X; the tangents at B, D meet at Y; prove $\angle AXC + \angle BYD = 2\angle AOD$.

27. PQ, PR are equal chords of a circle; PQ and the tangent at R intersect at T; prove $\angle PRQ = 60° \pm \frac{1}{3}\angle PTR$.

28. The diameter AB of a circle, centre O, is produced to T so that OB=BT; TP is a tangent to the circle; prove TP=PA.

29. The bisector of $\angle BAC$ cuts BC at D; a circle is drawn through D and to touch AC at A; prove that its centre lies on the perpendicular from D to AB.

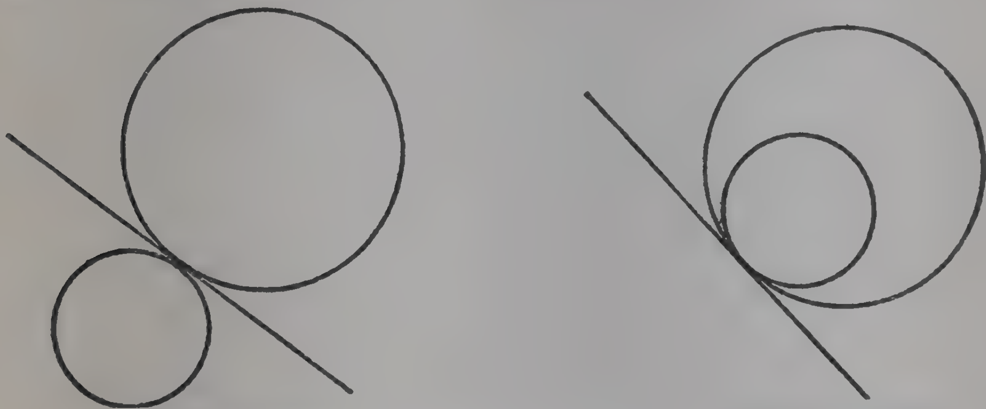
30. AB is a chord of a circle; the tangents at A, B meet at T; AP is drawn perpendicular to AB, and TP is drawn perpendicular to TA; prove that PT equals the radius.

31. Assuming the result of Exercise XXXIX. No. 21, what special cases can be obtained by taking (i) Q very close to S, (ii) Q very close to B, (iii) A very close to B?

32. OA is a chord of a circle, centre C; T is a point on the tangent at O such that OA=OT and $\angle AOT$ is acute; TA is produced to cut OC at B; prove that $\angle OBA = \frac{1}{2}\angle OCA$. Find the position of B when A is very close to O.

Definition.

If two circles touch the same line at the same point they are said to **touch each other** at that point. If the circles lie on

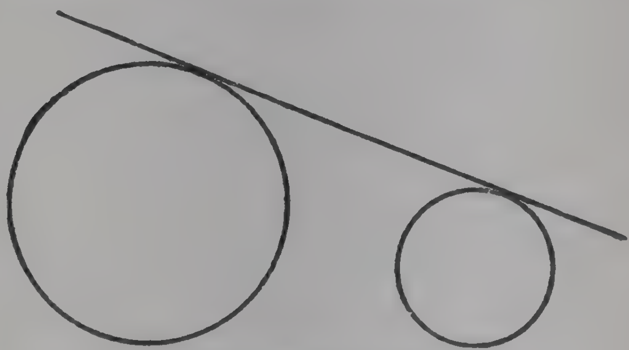


External contact. Internal contact.

Fig. 212.

opposite sides of the line they are said to touch **externally**; if they lie on the same side of the line they are said to touch **internally**.

If a straight line touches each of two circles it is called a **common tangent** to the two circles: it is called an **exterior common tangent** if the circles lie on the same side of it, and is called an **interior common tangent** if the circles lie on opposite sides of it.



Exterior common tangent.
Fig. 213 (1).

Interior common tangent.
Fig. 213 (2).

THEOREM 44.

If two tangents are drawn to a circle from an external point.

(1) The tangents are equal.

(2) The tangents subtend equal angles at the centre.

(3) The line joining the centre to the external point bisects the angle between the tangents.

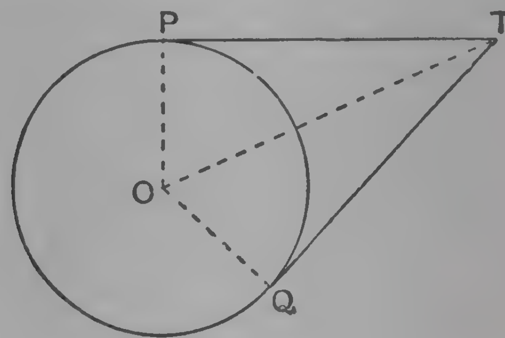


FIG. 214.

Given TP, TQ are the tangents from T to a circle, centre O.

To prove (1) $TP = TQ$.

(2) $\angle TOP = \angle TOQ$.

(3) $\angle OTP = \angle OTQ$.

Since TP, TQ are tangents at P, Q, the angles TPO, TQO are right angles.

$\therefore$ in the *right-angled* triangles TOP, TOQ

$OP = OQ$, radii.

OT is the common hypotenuse.

$\therefore$ $\triangle TOP \equiv \triangle TOQ$ (rt. angle, hyp., side).

$\therefore$ $TP = TQ$,

and $\angle TOP = \angle TOQ$,

and $\angle OTP = \angle OTQ$. Q.E.D.

THEOREM 45.

If two circles touch one another, the line joining their centres (produced if necessary) passes through the point of contact.

Given two circles, centres A, B, touching each other at P.

To prove AB (produced if necessary) passes through P.

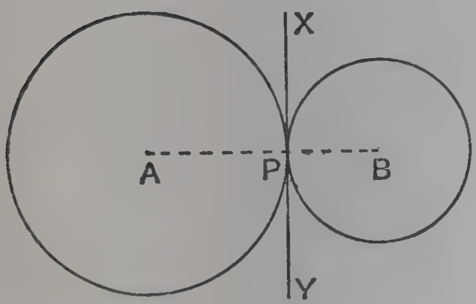


FIG. 215 (1).

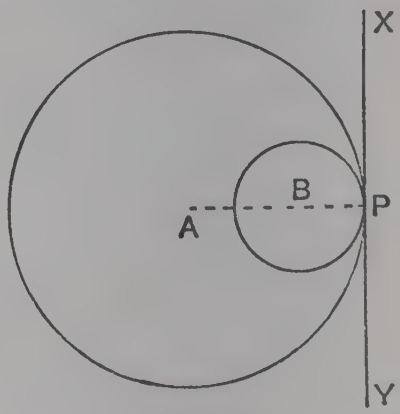


FIG. 215 (2).

Since the circles touch each other at P, they have a common tangent XPY at P.

Since XP touches each circle at P, the angles XPA, XPB are right angles.

∴ A and B each lie on the line through P perpendicular to PX.

∴ A, B, P lie on a straight line.

Q.E.D.

Note.—If two circles touch each other externally (Fig. 215 (1)), the distance between their centres equals the *sum* of the radii.

If two circles touch each other internally (Fig. 215 (2)), the distance between their centres equals the *difference* of the radii.

CONSTRUCTION 15.

(1) Construct a tangent to a circle at a given point on the circumference.

(2) Construct the tankents to a circle from a given point outside it.

(1) *Given* a point A on the circumference of a circle.

To construct the tangent at A to the circle.

Construct the centre O of the circle. Join AO.

Through A, construct a line AT perpendicular to AO.

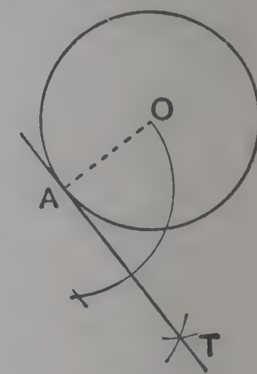


FIG. 216.

Then AT is the required tangent.

Proof. The tangent is perp. to the radius through the point of contact. But AO is a radius and $\angle OAT = 90°$,

$\therefore$ AT is the tangent at A.

(2) *Given* a point T outside a circle.

To construct the tangents from T to the circle.

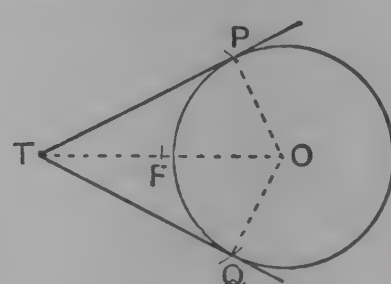


FIG. 217.

Construct the centre O of the circle. Join OT and bisect it at F. With centre F and radius FT, describe a circle and let it cut the given circle at P, Q. Join TP, TQ.

Then TP, TQ are the required tangents.

Proof. Since TF = FO, the circle, centre F, radius FT, passes through O, and TO is a diameter.

$\therefore \angle TPO = 90° = \angle TQO$. $\angle$ in semicircle.

But OP, OQ are radii of the given circle.

$\therefore$ TP, TQ are tangents to the given circle. Q.E.F.

CONSTRUCTION 16.

(1) Draw the direct (or exterior) common tangents to two circles.

(2) Draw the transverse (or interior) common tangents to two non-intersecting circles.

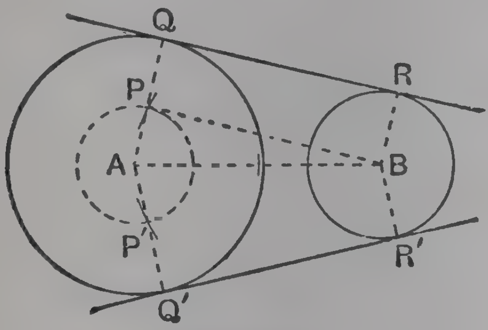


FIG. 218.

(1) *Given* two circles, centres A, B.

To construct their direct common tangents.

Let a, b be the radii of the circles, centres A, B, and suppose $a>b$. With A as centre and $a-b$ as radius, describe a circle and construct the tangents BP, BP′ from B to this circle. Join AP, AP′ and produce them to meet the circle, radius a, in Q, Q′. Through Q, Q′ draw lines QR, Q′R′ parallel to PB, P′B.

Then QR, Q′R′ are the required common tangents.

Proof. Draw BR, BR′ parallel to AQ, AQ′ to meet QR, Q′R′ at R, R′.

By construction, PQRB is a parallelogram.

$$\therefore \text{BR}=\text{PQ}=\text{AQ}-\text{AP}=a-(a-b)=b.$$

$\therefore$ R lies on the circle, centre B, radius b.

Also, since BP is a tangent, $\angle \text{BPA}=90°$.

$\therefore$ $\angle \text{RQA}=90°$ and $\angle \text{BRQ}=90°$, by parallels.

$\therefore$ QR is a tangent at Q and R to the two circles.

Similarly, it may be proved that Q′R′ is also a common tangent.

(2) *Given.* Two non-intersecting circles, centres A, B.

To construct the transverse common tangents.

Let a, b be the radii of the circles, centres A, B.

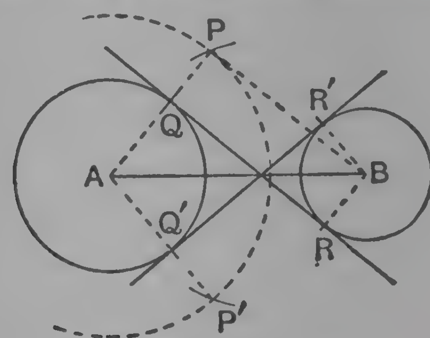


FIG. 219.

With A as centre and $a+b$ as radius, describe a circle and construct the tangents BP, BP′ to it from B.

Join AP, AP′, cutting the circle radius a at Q, Q′.

Through Q, Q′ draw lines QR, Q′R′ parallel to PB, P′B.

Then QR, Q′R′ are the required common tangents.

Proof. Through B draw BR, BR′ parallel to AQ, AQ′ to meet QR, Q′R′ at R, R′.

By construction, PBRQ is a parallelogram.

$\therefore$ BR = PQ = AP − AQ = $(a+b)-a=b$.

$\therefore$ R lies on the circle, centre B, radius b.

Also, since BP is a tangent, $\angle$BPA = 90°.

$\therefore$ $\angle$AQR = 90° and $\angle$BRQ = 90°, by parallels.

$\therefore$ QR is a tangent at Q and R to the two circles.

Similarly, it may be proved that Q′R′ is also a common tangent.

Q.E.F.

EXERCISE XLIII.

1. A circle, radius 5 cm., touches two concentric circles and encloses the smaller; the radius of the larger circle is 7 cm.; what is the radius of the smaller?

2. Three circles, centres A, B, C, touch each other externally; AB = 4″, BC = 6″, CA = 7″; find their radii.

3. In $\triangle$ABC, AB = 4″, BC = 7″, CA = 5″; two circles with B, C as centres touch each other externally; a circle with A as centre touches the others internally; find their radii.

4. Fig. 220 is formed of three circular arcs of radii 6·7 cm., 2·2 cm., 3·1 cm. ; X, Y, Z are the centres of the circles ; find the lengths of the sides of $\triangle XYZ$.

FIG. 220.

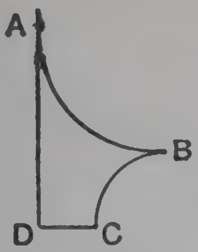


FIG. 221.

5. In Fig. 221, AB is a quadrant touching AD at A and the quadrant BC at B ; $\angle ADC = 90°$, $AD = 12''$, $DC = 9''$; find the radii of the circles.

6. The distance between the centres of two circles of radii 4 cm., 7 cm., is 15 cm. ; what is the radius of the least circle that can be drawn to touch them and enclose the smaller circle ?

7. C is a point on AB such that $AC = 5''$, $CB = 3''$; calculate the radius of the circle which touches AB at C and also touches the circle on AB as diameter.

8. A, B are the centres of two circles of radii 5 cm., 3 cm. ; $AB = 12$ cm. ; BC is a radius perpendicular to BA ; find the radius of a circle which touches the larger circle and touches the smaller circle at C. (Two answers.)

9. AB, BC are two equal quadrants touching at B ; their radii are 12 cm. ; find the radius of the circle which touches arc AB, arc BC, AC.

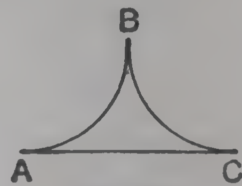


FIG. 222.

10. In $\triangle ABC$, $AB = 4''$, $BC = 6''$, $CA = 7''$; a circle touches BC, CA, AB at X, Y, Z ; find BX and AY.

11. In $\triangle ABC$, $AB = 3''$, $BC = 7''$, $CA = 9''$; a circle touches CA produced, CB produced, AB at Q, P, R ; find AQ, BR.

12. Two circles of radii 3 cm., 12 cm. touch each other externally ; find the length of their common tangent.

13. The distance between the centres of two circles of radii 11 cm., 5 cm. is 20 cm. ; find the lengths of their exterior and interior common tangents.

14. The distance between the centres of two circles is 10 cm., and the lengths of their exterior and interior common tangents are 8 cm., 6 cm. ; find their radii.

15. ABCD is a square of side $7''$; C is the centre of a circle of radius $3''$; find the radius of the circle which touches this circle and touches AB at A.

16. In one corner of a square frame, side $3'$, is placed a disc of radius $1'$ touching both sides; find the radius of the largest disc which will fit into the opposite corner.

17. a, b are the lengths of the diameters of two circles which touch each other externally; t is the length of their common tangent; prove that $t^2=ab$.

18. $OA=a''$, $OB=b''$, $\angle AOB=90°$; two variable circles are drawn touching each other externally, one of them touches OA at A, and the other touches OB at B; if their radii are x'', y'', prove that $(x+a)(y+b)$ is constant. If $a=8$, $b=6$, $x=4$, calculate y.

---

19. A circle touches the sides of $\triangle ABC$ at X, Y, Z; if Y, Z are the mid-points of AB, AC, prove that X is the mid-point of BC.

20. Two circles touch each other at A; any line through A cuts the circles at P, Q; prove that the tangents at P, Q are parallel.

21. ABCD is a quadrilateral circumscribing a circle; prove that $AB+CD=BC+AD$.

22. ABCD is a parallelogram; if the circles on AB and CD as diameters touch each other, prove that ABCD is a rhombus.

23. Two circles touch externally at A; PQ is their common tangent; prove that the tangent at A bisects PQ and that $\angle PAQ=90°$.

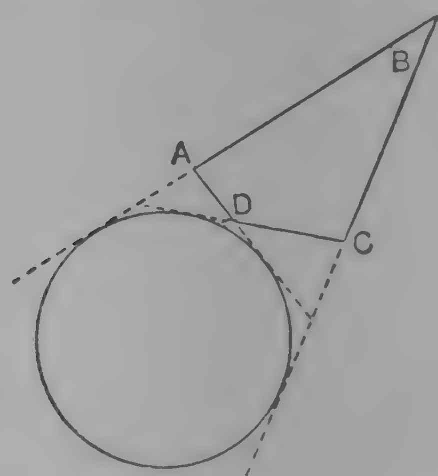


FIG. 223.

24. In Fig. 223, prove $AB-CD=BC-AD$.

25. ABCDEF is a hexagon circumscribing a circle; prove that $AB+CD+EF=BC+DE+FA$.

26. In $\triangle ABC$, $\angle BAC = 90°$; O is the mid-point of BC; circles are drawn with AB and AC as diameters; prove that two circles can be drawn with O as centre to touch each of these circles.

27. Two circles touch externally at A; AB is a diameter of one; BP is a tangent to the other; prove that $\angle APB = 45° - \frac{1}{2}\angle ABP$.

28. ABCD is a quadrilateral circumscribing a circle, centre O; prove $\angle AOB + \angle COD = 180°$.

29. Two circles touch internally at A; a chord PQ of one touches the other at R; prove $\angle PAR = \angle QAR$.

30. Two circles touch internally at A; any line PQRS cuts one at P, S and the other at Q, R; prove $\angle PAQ = \angle RAS$.

31. Two circles touch at A; any line PAQ cuts one circle at P, and the other at Q; prove that the tangent at P is perpendicular to the diameter through Q.

32. In $\triangle ABC$, $\angle ABC = 90°$; a circle, centre X, is drawn to touch AB produced, AC produced, and BC; prove $\angle AXC = 45°$.

33. Two circles touch externally at A; a tangent to one of them at P cuts the other circle at Q, R; prove $\angle PAQ + \angle PAR = 180°$.

34. Two circles, centres A, B, touch externally at P; a third circle, centre C, encloses both, touching the first at Q and the second at R; prove $\angle BAC = 2\angle PRQ$.

35. PR, QR are two circular arcs touching each other at R, and touching the unequal lines OP, OQ at P, Q; prove that $\angle PRQ = 180° - \frac{1}{2}\angle POQ$ (see Fig. 224).

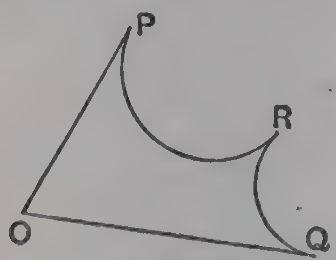


FIG. 224.

36. O is the centre of a fixed circle; two variable circles, centres P, Q touch the fixed circle internally and each other externally; prove that the perimeter of $\triangle OPQ$ is constant.

37. OA, OB are two radii of a circle, such that $\angle AOB = 60°$; a circle touches OA, OB and the arc AB; prove that its radius $= \frac{1}{3}$OA.

38. C is the mid-point of AB; semicircles are drawn with AC, CB, AB as diameters and on the same side of AB; a circle is drawn to touch the three semicircles; prove that its radius $= \frac{1}{3}$CA.

The converse of Theorem 43 is often of use in rider work. It may be stated as follows :

CONVERSE OF THEOREM 43.

If C and T are points on opposite sides of a line AB and such that $\angle BAT = \angle ACB$, then AT is a tangent to the circle which passes through A, C, B,

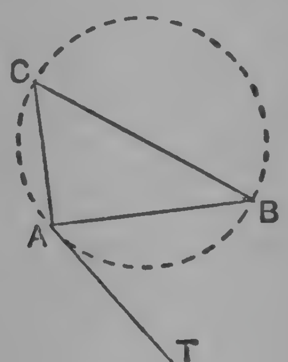


FIG. 225.

The next Exercise contains further examples on tests for concyclic points.

EXERCISE XLIV.

1. ABC is a $\triangle$ inscribed in a circle ; BE, CF are altitudes of $\triangle ABC$; prove that EF is parallel to the tangent at A.

2. ABCD is a parallelogram ; AC cuts BD at O ; prove that the circles AOB, COD touch each other.

3. A line AD is trisected at B, C ; BPC is an equilateral triangle ; prove that AP touches the circle PBD.

4. AB is a diameter, AP and AQ are two chords of a circle ; AP AQ cut the tangent at B in X, Y ; prove that P, X, Y, Q are concyclic.

5. ABC is a $\triangle$ inscribed in a circle ; any line parallel to AC cuts BC at X, and the tangent at A at Y ; prove B, X, A, Y are concyclic.

6. In $\triangle ABC$, $\angle BAC = 90°$; the perpendicular bisector of BC cuts CA, BA produced at P, Q ; prove that BC touches the circle CPQ.

7. ABCDE is a regular pentagon ; BD cuts CE at O ; prove that BC touches the circle BOE.

8. OY is the bisector of $\angle XOZ$; P is any point; PX, PY, PZ are the perpendiculars to OX, OY, OZ; prove that $XY = YZ$.

9. CA, CB are two fixed radii of a circle; P is a variable point on the circumference; PQ, PR are the perpendiculars from P to CA, CB; prove that QR is of constant length.

10. ABC is a $\triangle$ inscribed in a circle; a line parallel to AC cuts BC at P, and the tangent at A at T; prove that $\angle APC = \angle BTA$.

11. Four circular coins of unequal sizes lie on a table so that each touches two, and only two, of the others; prove that the four points of contact are concyclic.

12. ABC, ABD are two equal circles; if $AB = BC$, prove that AC touches the circle ABD.

13. AC, BD are two perpendicular chords of a circle; prove that the tangents at A, B, C, D form a cyclic quadrilateral.

14. AB, AC are two equal chords of a circle; AP, AQ are two chords cutting BC at X, Y; prove P, Q, X, Y are concyclic.

15. ABC is a $\triangle$ inscribed in a circle; AD is an altitude of $\triangle ABC$; DP is drawn parallel to AB and meets the tangent at A at P; prove $\angle CPA = 90°$.

16. The side CD of the square ABCD is produced to E; P is any point on CD; the line from P perpendicular to PB cuts the bisector of $\angle ADE$ at Q; prove $BP = PQ$.

17. PQ, CD are parallel chords of a circle; the tangent at D cuts PQ at T; B is the point of contact of the other tangent from T; prove that BC bisects PQ.

MENSURATION.

1. For a *circle* of radius r inches,
   (i) the length of the circumference $= 2\pi r$ in.
   (ii) the area of the circle $= \pi r^2$ sq. in.
   (iii) the length of an arc which subtends $\theta°$ at the centre of the circle, $= \dfrac{\theta}{360} \times 2\pi r$ in.
   (iv) the area of a sector of a circle of angle $\theta° = \dfrac{\theta}{360} \times \pi r^2$ sq. in.

2. For a *sphere* of radius r inches,
   (i) the area of surface of sphere $= 4\pi r^2$ sq. in.
   (ii) the volume of the sphere $= \frac{4}{3}\pi r^3$ cub. in.
   (iii) the area of the surface intercepted between two parallel planes at distance d inches apart $= 2\pi rd$ sq. in.

3. For a *circular cylinder*, radius r inches, height h inches,
   (i) the area of the curved surface $= 2\pi rh$ sq. in.
   (ii) the volume of the cylinder $= \pi r^2 h$ cub. in.

4. For a *circular cone*, radius of base r inches, height h inches length of slant edge l inches,
   (i) $l^2 = r^2 + h^2$.
   (ii) area of the curved surface $= \pi rl$ sq. in.
   (iii) volume of cone $= \frac{1}{3}\pi r^2 h$ cub. in.

5. (i) The volume of any cylinder = area of base × height.
   (ii) The volume of any pyramid = $\frac{1}{3}$ area of base × height.
   $\pi = \frac{22}{7}$ approx. or 3·1416 approx.

EXERCISE XLV.

1. Find (1) the circumference, (2) the area of a circle of radius (i) 4″, (ii) 100 yards.

2. The circumference of a circle is 5 inches; what is its radius correct to $\frac{1}{10}$ inch?

3. The area of a circle is 4 sq. cm.; what is its radius correct to $\frac{1}{10}$ cm.?

4. An arc of a circle of radius 3 inches subtends an angle of 40° at the centre; what is its length correct to $\frac{1}{10}$ inch?

5. The angle of a sector of a circle is 108°, and its radius is 2·5 cm.; what is its area?

6. A square ABCD is inscribed in a circle of radius 4 inches; what is the area of the minor segment cut off by AB.

7. AB is an arc of a circle, centre O; AO=5 cm. and arc AB=5 cm.; find ∠AOB, correct to nearest minute.

8. A piece of flexible wire is in the form of an arc of a circle of radius 4·8 cm. and subtends an angle of 240° at the centre of the circle: it is bent into a complete circle; what is the radius?

9. A horse is tethered by a rope 5 yards long to a ring which can slide along a low straight rail 8 yards long; what is the area over which the horse can graze?

10. OA, OB are two radii of a circle; prove that the area of sector AOB equals $\frac{1}{2}$OA X arc AB.

11. What is the area contained between two concentric circles of radii 6 inches, 3 inches?

12. In Fig. 226, AB, BC, CD, DA are quadrants of equal circles of radii 5 cm., touching each other. Find the area of the figure.

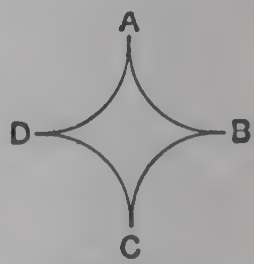


FIG. 226.

13. Find (i) the volume, (ii) the *total* surface of a closed cylinder, height 8″, radius 5″.

14. 1 lb. of tobacco is packed in a cylindrical tin of diameter 4″ and height 8″; what would be the height of a tin of diameter 3″ which would hold $\frac{1}{4}$ lb. of tobacco, similarly packed?

15. How many cylindrical glasses 2″ in diameter can be filled to a depth of 3″ from a cylindrical jug of diameter 5″ and height 12″?

16. Find (i) the volume, (ii) the area of the curved surface of a circular cone, radius of base 5″, height 12″.

17. A sector of a circle of radius 5 cm. and angle 60° is bent to form the surface of a cone; find the radius of its base,

18. The curved surface of a circular cone, height 3″, radius of base 4″ is folded out flat. What is the angle of the sector so obtained?

19. Find (i) the volume, (ii) the *total* area of the surface of a pyramid, whose base is a square of side 6″ and whose height is 4″.

20. Find (i) the volume, (ii) the area of the surface of a sphere of diameter 5 cm.

21. Taking the radius of the earth as 4000 miles, find the area between latitudes 30° N. and 30° S. What fraction is this area of the area of the total surface of the earth?

22. Two cylinders, diameters 8″ and 6″, are filled with water to depths 10″, 5″ respectively; they are connected at the bottom by a tube with a tap; when the tap is turned on, what is the resulting depth in each cylinder?

23. Three draughts, $1\frac{1}{2}$″ in diameter, are placed flat on a table and an elastic band is put round them. Find its stretched length.

24. What is the length of a belt which passes round two wheels of diameters 2″, 4″, so that the two straight portions cross at right angles? (see Fig. 227).

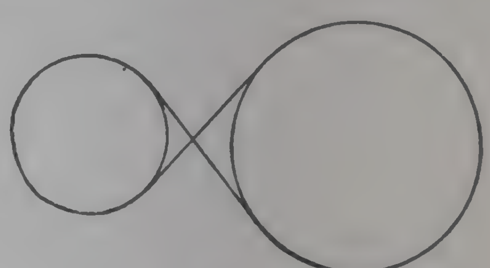


FIG. 227.

25. A circular metal disc, 9″ in diameter, weighs 6 lb.; what is the weight of a disc of the same metal, 6″ in diameter and of the same thickness?

26. Find the volume of the greatest circular cylinder that can be cut from a rectangular block whose edges are 4″, 5″, 6″.

27. Draw a circle of radius 5 cm. and place in it a chord AB of length 4 cm.; find the area of the major segment AB, making any measurements you like.

28. A rectangular lawn 15 yards by 10 yards is surrounded by flower-beds: a man can, without stepping off the lawn, water the ground within a distance of 5 feet from the edge. What is the total area of the beds he can so water?

What would be the area within his reach, if the lawn was in the shape of (i) a scalene triangle, (ii) any convex polygon, of perimeter 50 yards?

29. ABC is a right-angled triangle; circles are drawn with AB, BC, CA as diameters; prove that the area of the largest is equal to the sum of the areas of the other two circles.

30. Fig. 228 represents four semicircles; AC=DB and XOV bisects AB at right angles. Prove that

(i) Curves AXB AVB are of equal lengths;

(ii) Area of figure=area of circle on XV as diameter.

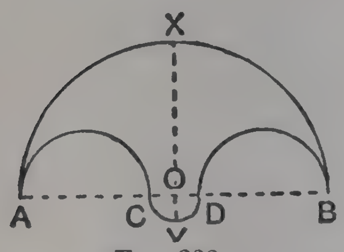


FIG. 228.

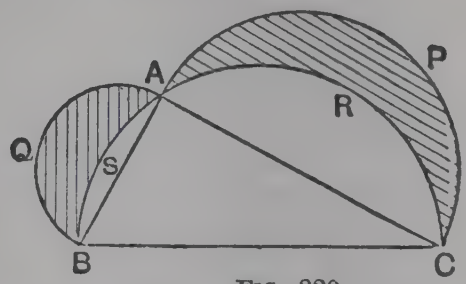


FIG. 229.

31. In Fig. 229, BQA, APC, BSARC are semicircles, prove that the sum of the areas of the lunes BSAQ, CRAP equals the area of $\triangle$ABC.

32. A bowl (see Fig. 230) consists of a hollow sphere of radius 20 inches with a small hole at the top A; a *solid* sphere of radius 15 inches is fixed inside it so that the two spheres are concentric. Water is poured into the bowl through A at a steady rate of 10 cubic inches per second. Find the rate at which the water level is rising when the inner sphere is just half under water, and show that the water level continues to rise at the same rate until the inner sphere is completely covered.

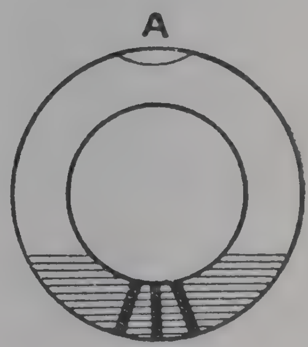


FIG. 230.

33. The diagonals AC, BD of the quadrilateral ABCD cut at right angles at O; AO=6″, OC=OD=2″, OB=4″. The triangle DOC is cut away and the triangles AOD, BOC are folded through 90° about OA, OB so as to form two faces of a tetrahedron on $\triangle$OAB as base.

Find (i) the volume of the tetrahedron;

(ii) the area of the remaining face;

(iii) the length of the perpendicular from O to the opposite face.

34. ABCD is a rectangle; AB=10″, AD=6″; AXB, BYC, CZD, DWA are isosceles triangles, all the equal sides of which are 9″; they are folded so as to form a pyramid with ABCD as base and X, Y, Z, W at the vertex.

Find (i) the height of the pyramid;

(ii) the volume of the pyramid;

(iii) the *total* area of the surface of the pyramid.

If AB=p'', AD=q'', AX=r'', and if the height of the pyramid=h'', prove that $h^2=r^2-\frac{1}{4}p^2-\frac{1}{4}q^2$.

CONSTRUCTION OF CIRCLES.

CONSTRUCTION 17.

(1) Construct the inscribed circle of a given triangle.

(2) Construct an escribed circle of a given triangle.

Given a triangle ABC.

To construct (1) the circle inscribed in $\triangle$ABC;

(2) the circle which touches AB produced, AC produced and BC.

(1) Construct the lines BI, CI, bisecting the angles ABC, ACB and intersecting at I.

Draw IX perpendicular to BC.

With I as centre and IX as radius, describe a circle.

This circle touches BC, CA, AB.

Proof. Since I lies on the bisector of $\angle$ABC.

I is equidistant from the lines BA, BC.

Since I lies on the bisector of $\angle$ACB.

I is equidistant from the lines CB, CA.

$\therefore$ I is equidistant from AB, BC, CA.

$\therefore$ the circle, centre I, radius IX, touches AB, BC, CA.

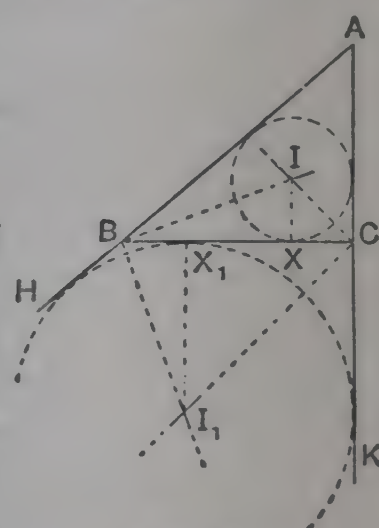


FIG. 231.

(2) Produce AB, AC to H, K. Construct the lines BI_1, CI_1, bisecting the angles HBC, KCB and intersecting at I_1.

Draw I_1X_1 perpendicular to BC.

With I_1 as centre and I_1X_1 as radius, describe a circle.

This circle touches AB produced, AC produced and BC.

Proof. Since I_1 lies on the bisector of $\angle$HBC.

I_1 is equidistant from BH and BC.

Since I_1 lies on the bisector of $\angle$KCB,

I_1 is equidistant from CK and CB.

$\therefore$ I_1 is equidistant from HB, BC, CK.

$\therefore$ the circle, centre I_1, radius I_1X_1, touches HB, BC, CK.

Q.E.F.

CONSTRUCTION 18.

On a given straight line, construct a segment of a circle containing an angle equal to a given angle.

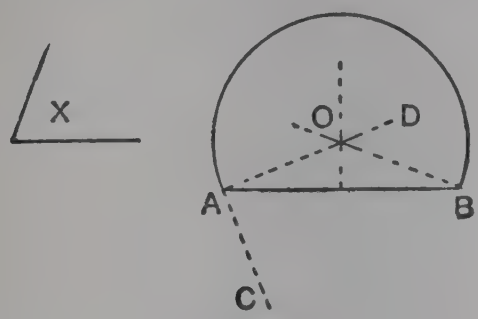


FIG. 232.

Given a straight line AB and an angle X.

To construct on AB a segment of a circle containing an angle equal to $\angle X$.

At A, make an angle BAC equal to $\angle X$.

Draw AD perpendicular to AC.

Draw the perpendicular bisector of AB and let it meet AD at O.

With O as centre and OA as radius, describe a circle.

Then the segment of this circle on the side of AB opposite to C is the required segment.

Proof. Since O lies on the perpendicular bisector of AB, $OA = OB$; $\therefore$ the circle passes through B.

Since AC is perpendicular to the radius OA, AC is a tangent;

$$\therefore \angle X = \angle CAB = \text{angle in alternate segment.}$$

Q.E.F.

CONSTRUCTION 19.

Construct a circle to pass through a given point A and to touch a given circle at a given point B.

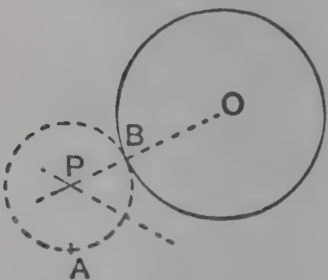


FIG. 233.

Construct the centre O of the given circle.

Construct the perpendicular bisector of AB and produce it to cut OB, or OB produced at P.

With P as centre and PB as radius, describe a circle. This is the required circle.

Proof. Since P lies on the perpendicular bisector of AB, AP = PB.

Since P lies on OB, or OB produced, the two circles touch at B.

Q.E.F.

Inscribed and Circumscribed Regular Polygons.

If a regular polygon of n sides is inscribed in a circle or circumscribed about a circle, each side subtends an angle of $\frac{360}{n}$ degrees at the centre of the circle.

For the values $n=3, 4, 6, 8$, these angles are respectively $120°$, $90°$, $60°$, $45°$, and angles of these magnitudes can be constructed by ruler and compass without using a protractor.

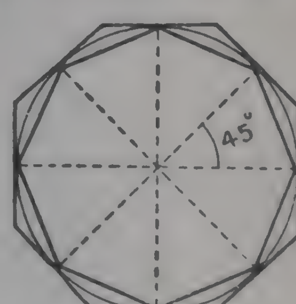


FIG. 234.

Fig. 234 represents a regular octagon inscribed in a circle and circumscribed about a circle.

The reader should construct inscribed and circumscribed figures of 3, 4 and 6 sides.

The simplest way of inscribing a regular hexagon in a circle is to make use of the fact that the length of each side is equal to the radius. To circumscribe a regular hexagon about a circle, draw tangents at the corners of the inscribed regular hexagon.

EXERCISE XLVI.

1. Use a coin to draw a circle, and construct its centre.

2. Given two points A, B, and a line CD, construct a circle to pass through A and B and have its centre on CD.

3. Draw a line AB 3 cm. long, and construct a circle of radius 5 cm. to pass through A and B.

4. Draw two lines AOB, COD intersecting at an angle of 80°, make AO=3 cm., OB=4 cm., CO=5 cm., OD=2·4 cm., construct a circle to pass through A, B, C. Does it pass through D ?

5. Construct two circles of radii 4 cm., 5 cm., such that their common chord is of length 6 cm. Measure the distance between their centres.

6. Draw two lines OAB, OCD intersecting at an angle of 40°; make OA=2 cm., OB=6 cm., OC=3 cm., OD=4 cm.; construct a circle to pass through A, B, C. Does it pass through D ?

7. Given a circle and two points A, B inside it, construct a circle to pass through A and B and have its centre on the given circle.

8. Given a point B on a given line ABC and a point D outside the line, construct a circle to pass through D and to touch AC at B.

9. Draw a line AB and take a point C at a distance of 3 cm. from the line AB; construct a circle of radius 4 cm. to pass through C and touch AB.

10. Draw two lines AB, AC making an angle of 65° with each other; construct a circle of radius 3 cm. to touch AB and AC.

11. Draw a circle of radius 3 cm. and take a point A at a distance of 4 cm. from its centre; construct a circle to touch the first circle and to pass through A, and to have a radius of 2 cm. Is there more than one such circle ?

12. Given a straight line and a circle, construct a circle of given radius to touch both the straight line and the circle. Is this always possible ? If not, state the conditions under which it is impossible.

13. Draw a line AB of length 6 cm.. with A, B as centres and radii 3 cm., 2 cm. respectively, describe circles. Construct a circle to touch each of these circles and have a radius of 5 cm. Give all possible solutions. (The contacts may be internal or external.)

14. Draw a circle of radius 4·5 cm., and draw a diameter AB, construct a circle of radius 1·5 cm. to touch the circle and AB.

15. Given a circle and a point A on the circle and a point B outside the circle, construct a circle to pass through B and to touch the given circle at A.

16. Draw a circle of radius 5 cm.; construct two circles of radii 1·5 cm., 2·5 cm. touching each other externally and touching the first circle internally.

17. Draw a triangle whose sides are of lengths 2, 3, 4 cm., and construct the four circles which touch the sides of this triangle, and measure their radii.

18. Draw two lines OA, OB such that $\angle AOB = 40°$, and $OA = 4$ cm.; construct a circle touching OA at A and touching OB; measure its radius.

19. Given a triangle ABC, construct a circle to touch AB, AC and have its centre on BC. Is there more than one solution ?

20. Inscribe a circle in a given sector of a circle. (*i.e.* Given two radii OA, OB of a circle, construct a circle to touch OA, OB and the arc AB.)

21. Given two points A, B and a point D on a line CDE, construct two concentric circles one of which passes through A, B and the other touches CE at D. When is this impossible ?

22. Given three points, A, B, C, construct three circles with these points as centres so that each circle touches the other two. Is there more than one solution ?

23. Given two points A, B, 4 cm. apart, construct a circle to pass through A and B and such that the tangents at A and B include an angle of 100°; measure its radius.

24. ABC is a triangle such that $BC = 6$ cm., $BA = 4$ cm., $\angle ABC = 90°$; find by measurement the radius of the circle escribed to BC.

25. Given two parallel lines and a point between them, construct a circle to touch the given lines and pass through the given point.

26. Draw a quadrilateral so that its sides in order are 4, 5, 7, 6 cm.; inscribe a circle in it to touch three of the sides. Does it touch the fourth side ?

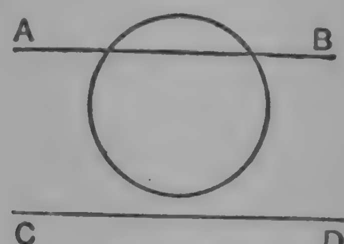


FIG. 235.

27. In Fig. 235, AB, CD are two given parallel lines: construct a circle to touch AB, CD and the given circle.

28. Given two circles, centres A, B, radii a, b, and a point C on the first, construct a circle to touch the first circle at C and also to touch the second. Fig. 236 gives the construction for the centre P of the required circle, if it touches both circles externally. D is found by making $CD=b$. Perform this construction and construct also the circle in the case where the contacts are external with circle A, internal with circle B. How would D be situated if the constructed circle touches circle A internally and circle B either internally or externally ?

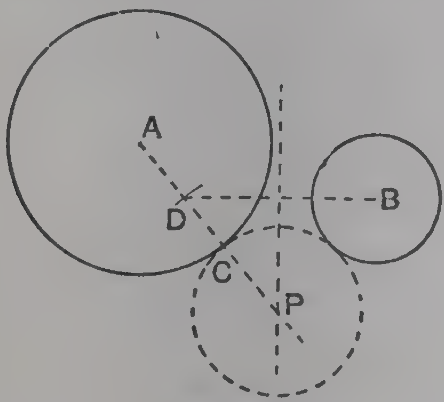


FIG. 236.

29. ABC is an equilateral triangle ; AB=4 cm. ; A, B are the centres of two equal circles of radii 2·5 cm. ; CA is *produced* to meet the first circle at D. Construct a circle touching the first circle internally at D and touching the second circle externally. State your construction.

30. Construct a circle to touch a given line AB and a given circle centre C, at a given point D. Fig. 237 gives the construction for the centre P of the required circle if the contact is external. Perform the construction and construct the case where the contact is internal.

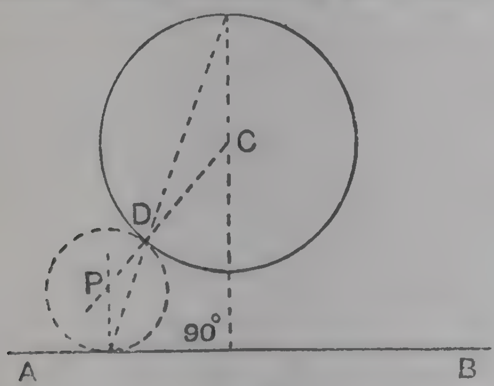


FIG. 237.

Construct the Figs. in Exs. 31-47 : **do not rub out any of your construction lines.**

31. Three arcs each of radius 3 cm. and each $\frac{1}{6}$th of a complete circumference (Fig. 238).

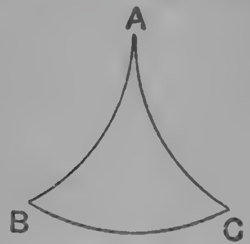


FIG. 238.

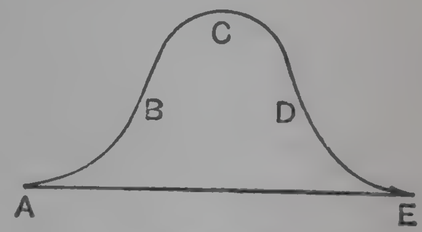


FIG. 239.

32. AB, BC, CD, DE are equal quadrants ; AE=6 cms.

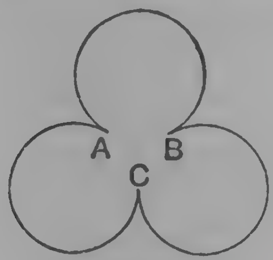


FIG. 240.

33. Three arcs each of radius 3 cm. touching at A, B, C.

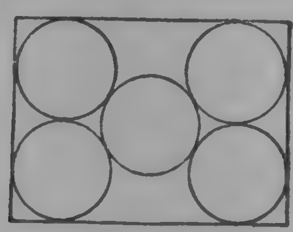


FIG. 241.

34. The sides of the rectangle are 6 cm., 8 cm., and the four outer circles are equal.

35. The radii of the arcs AB, BC, CA are 3·5 cm., 2·5 cm., 7 cm.

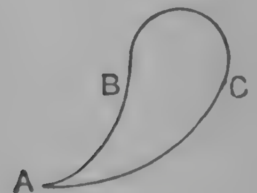


FIG. 242.

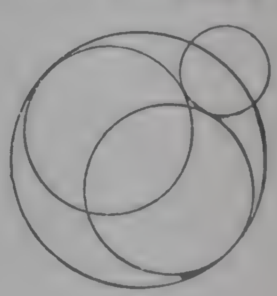


FIG. 243.

36. The radii of the circles are 1 cm., 2 cm., 2 cm., 3 cm., and the centre of the smallest circle lies on the largest (Fig. 243).

37. AP, AQ are arcs of radii 4 cm.; PQ is of radius 8 cm.; AB is perpendicular to CD and equals 3 cm.

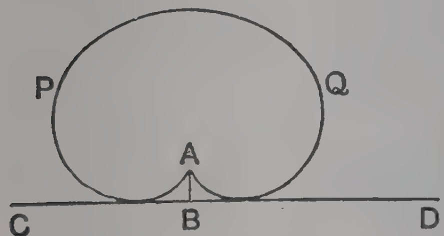


FIG. 244.

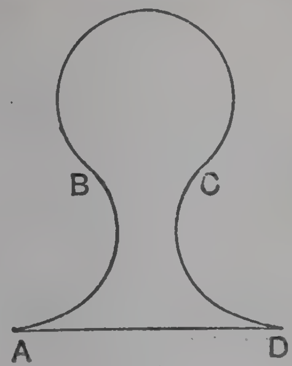


FIG. 245.

38. AB, BC, CD are arcs of radii 3 cm., AD equals 7 cm. and touches AB, DC.

39. The radii of the arcs AB, BC are 3·5 cm., 1·2 cm., CD=5 cm., DE=6·5 cm., AE=7 cm. (Fig. 246).

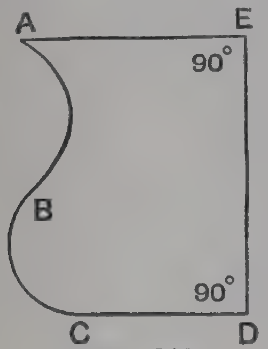


FIG. 246.

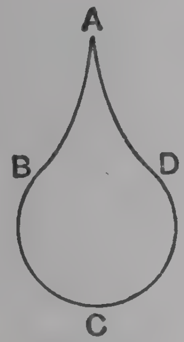


FIG. 247.

40. AB, AD are arcs of radii 6 cm.; AC equals 6 cm. and is an axis of symmetry (Fig. 247).

41. CE is an axis of symmetry; AB, BC are arcs each of radius 3 cm.; the centre of AB lies on AD. AD=10 cm., CE=5 cm. (Fig. 248).

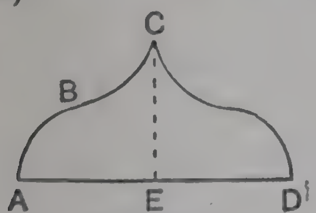


FIG. 248.

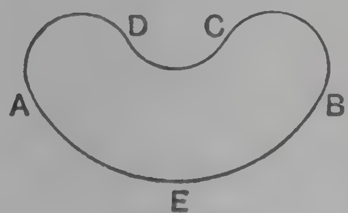


FIG. 249.

42. AB is an arc of radius 3 cm.; BC, CD, DA are arcs each of radius 1 cm.; chord AE=chord EB=3 cm. (Fig. 249).

43. CD is an axis of symmetry; AB=9·5 cm., CD=3·5 cm.; AE, EC are arcs of radii 2, 10·5 cm. respectively.

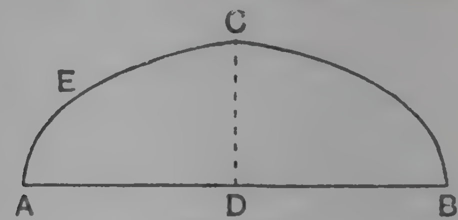


FIG. 250.

44. AB is a quadrant of radius 2·5 cm. with its centre on AC; AC=7 cm. The arc BC touches AB at B.

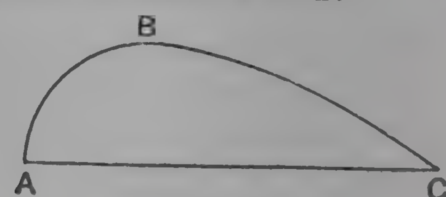


FIG. 251.

45. ABCD is a square of side 2 cm.; BE, EF are circular arcs with C, A as centres respectively.

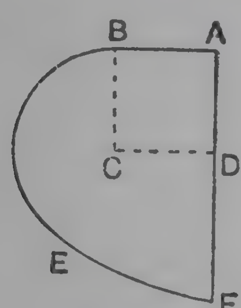


FIG. 252.

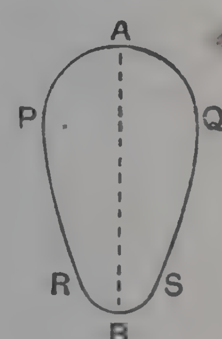


FIG. 253.

46. AB is an axis of symmetry; PAQ is a semicircle of radius 2 cm.; RBS is an arc of radius 1 cm.; AB=7 cm. The arcs PR, QS are tangential at each end.

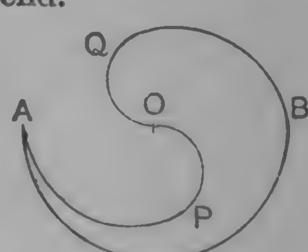


FIG. 254.

47. AB is a semicircle, radius 3 cm., centre O; OP, OQ are arcs each of radius 1 cm.; the arcs AP, AB are tangential at A.

MISCELLANEOUS CONSTRUCTIONS. III.

EXERCISE XLVII.

1. Draw a circle of radius 3 cm., and construct a chord of the circle of length 5 cm. Take a point A inside the circle but not on the chord, and construct a chord of length 5 cm. passing through A.

2. Given a chord PQ of a given circle and a point R on PQ, construct a chord through R equal to PQ.

3. Inscribe a regular hexagon in a given circle.

4. Inscribe an equilateral triangle in a given circle.

5. A, B, C are three given points on a given circle; construct a chord of the circle equal to AB and parallel to the tangent at C.

6. Draw a circle radius 4 cm. and take a point 6 cm. from the centre. Construct the tangents from this point to the circle and measure their lengths.

7. Draw a circle of radius 3 cm., and construct two tangents which include an angle of 100°.

8. Draw a line AB of length 7 cm.; construct a line AP such that the perpendicular from B to AP is 5 cm.

9. Draw a circle, centre O, radius 4 cm.; take a point A 6 cm. from O; draw AB perpendicular to AO; construct a point P on AB such that the tangent from P to the circle is of length 5·5 cm.; measure AP.

10. Draw a circle of radius 3 cm. and take a point 5 cm. from the centre; construct a chord of the circle of length 4 cm. which when produced passes through this point.

11. Draw a line AB of length 5 cm. and describe a circle with AB as diameter; construct a point on AB produced such that the tangent from it to the circle is of length 3 cm.

12. Given a circle and a straight line, construct a point on the line such that the tangents from it to the circle contain an angle equal to a given angle.

13. Circumscribe an equilateral triangle about a given circle.

14. On a line of length 5 cm., construct a segment of a circle containing an angle of 70°; measure its radius.

15. On a line of length 2 inches, construct a segment of a circle containing an angle of 140°; measure its radius.

16. In a circle of radius 3 cm., inscribe a triangle whose angles are 40°, 65°, 75°; measure its longest side.

17. Inscribe in a circle of radius 1″ a rectangle of length 1·5″, and measure its breadth.

18. Circumscribe about a circle of radius 2 cm. a triangle whose angles are 50°, 55°, 75°; measure its longest side.

19. Draw two circles of radii 2 cm., 3 cm., with their centres 6·5 cm. apart; construct their four common tangents.

20. Draw two circles of radii 2·5 cm., 3·5 cm., touching each other externally, and construct their exterior common tangents.

21. Draw a line AB of length 6 cm. and construct a line PQ such that the perpendiculars to it from A, B are of lengths 2 cm., 4 cm.

22. Draw two circles of radii 2 cm., 3 cm., with their centres 6 cm. apart; construct a chord of the larger circle of length 4 cm. which when produced touches the smaller circle.

23. Construct the triangle ABC, given that BC=6 cm., ∠BAC =90°, the altitude AD=2 cm.; measure AB, AC.

24. Construct the triangle ABC, given that BC=5 cm., ∠BAC =55°, the altitude AD=4 cm.; measure AB, AC.

25. Construct a triangle ABC, given BC=6 cm., ∠BAC=52°, and the median BE=5 cm.

26. Draw a circle of radius 3 cm., and construct points A, B, C on the circumference such that BC=5 cm., BA+AC=8·1 cm.; measure BA and AC.

27. Construct a triangle ABC given its perimeter, the angle BAC and the length of the altitude AD.

28. Draw any circle and take two points A, B on it and a point C outside the circle; construct a point P on the circle such that PC bisects ∠APB.

29. Draw two lines which meet at a point off your paper; construct the bisector of the angle between them.

30. Construct the quadrilateral ABCD, given that AD=5 cm., BC=4·6 cm., ∠ABD=∠ACD=55°, ∠CBD=43°; measure CD.

31. Draw any circle and take two points A, B on it; construct a point P on the circle such that chord PA equals twice chord PB.

32. Draw two unequal circles intersecting at A, B; construct a line through A, cutting the circles at P, Q such that PQ is of given length.

33. Circumscribe a square about a given quadrilateral.

REVISION PAPERS.

SECTIONS I.-IV.

37.

1. ABCD is a square; the bisector of $\angle ACD$ cuts BD at Q; prove that $BQ=CD$.

2. AD, BC are the parallel sides of the trapezium ABCD; $AB=6''$, $BC=9''$, $CD=5''$, $AD=14''$; find the area of ABCD.

3. O is the centre of a circle; CB is a chord parallel to a radius OA; OB cuts AC at a point K inside the circle; prove $\angle AKB = 3\angle ACB$.

4. A halfpenny (diameter 1″) rolls once completely round the outside of a rectangle 3″ by 4″. What is the distance travelled by its centre?

38.

1. In the $\triangle ABC$, $\angle ABC=54°$, $\angle BAC=78°$; the bisector of $\angle BCA$ cuts AB at X; prove that $CA=CX$.

2. A segment of a circle is cut off by a chord of length 6 cm.; the height of the segment is 2 cm. Calculate the radius of the circle.

3. AB is a quadrant of a circle, AC is any chord; BN is the perpendicular from B to AC; prove that $BN=NC$.

4. AB is a chord of a circle; AT is the tangent at A; AC is a chord bisecting $\angle BAT$; prove that $AC=CB$.

39.

1. ABCD is a parallelogram; $AB<BC$; the line bisecting $\angle ABC$ cuts AD at P; prove that $BC=CD+DP$.

2. Fig. 255 represents two equal circular arcs whose centres lie on AB. Given $AB=10$ cm., $CD=4$ cm., calculate the radius of each arc.

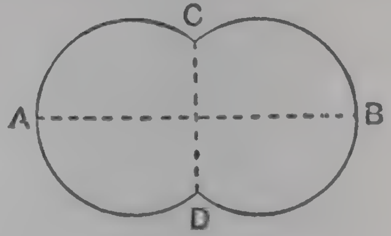


FIG. 255.

3. ABCD is a quadrilateral inscribed in a circle centre O; $\angle ABD=36°$,

$$\angle BDC=43°,\ \angle COD=40°;$$

calculate $\angle AOB$.

4. ABCDE is a pentagon inscribed in a circle; AT is the tangent at A, T and D being on opposite sides of AB; prove that $\angle BCD + \angle AED = 180° + \angle BAT$.

40.

1. ABCD is a quadrilateral such that $\angle BAD = \angle BCD = 90°$, prove that the bisectors of $\angle$s ADC, ABC are parallel.

2. AP, BQ, CR, DS are four equal parallel edges of a box with rectangular faces; ABCD and PQRS are the two end faces. $AB = 3''$, $BC = 2''$, $AP = 5''$; a piece of string fastened to A is wound round the box, crossing in turn the edges BQ, CR, DS and ending at P. Find its length when taut. (Cut down the edge AP, remove the end faces, and fold the sides of the box out flat.)

3. The triangle ABC is obtuse-angled at A; CN is the perpendicular from C to BA produced; O is the centre of the circle through A, B, C; prove that $\angle ACN = \angle OCB$.

4. AB is a diameter of a circle, AC is any chord; P is the mid-point of the arc BC; prove that AC is perpendicular to the tangent at P.

41.

1. ABCD is a square; any line is drawn through A outside the square; BH, DK are the perpendiculars from B, D to this line; prove that

(i) $\triangle ABH \equiv \triangle DAK$;

(ii) $BH + DK = HK$.

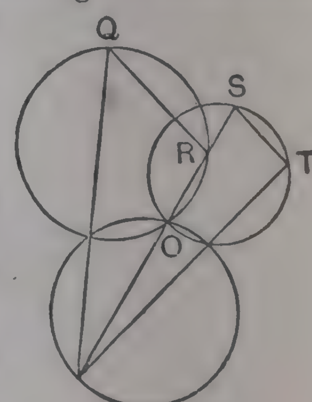


FIG. 256.

2. In the quadrilateral ABCD, $AB = 5''$, $BC = 12''$, $CD = 7''$, $\angle ABC = \angle BCD = 90°$; P is a point on BC such that $\angle APD = 90°$; calculate the length of BP. (Two answers.)

3. In Fig. 256, prove that QR is parallel to ST.

4. A circle is drawn touching the sides BC, CA, AB of the triangle ABC at X, Y, Z. If $\angle ABC = 36°$, $\angle ACB = 66°$, calculate $\angle YXZ$.

42.

1. ABC is an equilateral triangle; P, Q, R are points on BC, CA, AB such that PQR is also equilateral; prove that $AQ + AR = BC$.

2. ABC is a triangle such that $\angle BAC = 7\angle ABC$; prove that it is possible to find points P, Q on BC, such that each of the triangles APB, AQP, ACQ is isosceles.

3. ABC is an equilateral triangle; the circle on BC as diameter cuts BA at R; P is the mid-point of BC; prove that $BR = BP$.

4. With the data of question 3, calculate the area of the part of the circle which lies inside the triangle, given $BC = 4''$.

43.

1. ABCD is a parallelogram BCH, DCK are equilateral triangles outside it; prove that (i) $\triangle ADK \equiv \triangle HBA$; (ii) if AD is produced to E, $\angle EDK = \angle DAB - \angle KAH$; (iii) $\angle KAH = 60°$; (iv) KA=KH.

2. The sides of a triangle are 2·8″, 4·5″, 5·4″. Is it an obtuse angled triangle or not ?

3. ABCD is a quadrilateral, right-angled at B, C; a line perpendicular to AD cuts AD, BC at P, Q; prove that $\angle BPC = \angle AQD$.

4. A, B, C are three points on a circle, A being on the major arc BC; the tangents at B, C intersect at T; a line is drawn through T parallel to the tangent at A, and cuts AB, AC produced at P, Q; prove that TP=TQ.

44.

1. In the quadrilateral ABCD, $\angle DAB = \angle ABC = 60°$, $\angle ADC = 90°$; prove that AB+BC=2AD.

2. In Fig. 257, PN is perpendicular to AC and PR is parallel to AC; also QR=2AP; prove that $\angle CAR = \frac{1}{3}\angle CAB$. (Join P to the mid-point of QR.)

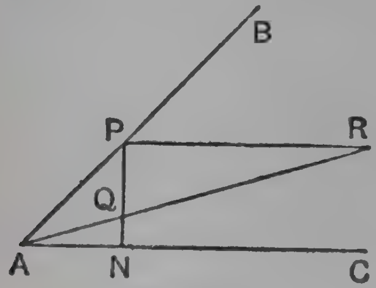


FIG. 257.

3. The radii of two circles are 2 cm., 5 cm., and the distance between their centres is 9 cm.; calculate the lengths of the internal and external common tangents.

4. AB is a diameter of a circle, centre O; the tangent at B meets any chord AP when produced at T; prove that $\angle AOP = 2\angle ATB$.

45.

1. The side BC of an equilateral triangle ABC is produced to D so that CD=3BC; prove $AD^2 = 13AB^2$.

2. ABCD is a quadrilateral; if $\angle ABC + \angle ADC = 180°$, prove that the perpendicular bisectors of AC, BD, AB are concurrent.

3. ABCD is a quadrilateral inscribed in a circle; AC is a diameter; $\angle BAC = 43°$; find $\angle ADB$.

4. Two circles ABPQ, ABR intersect at A, B; BP is a tangent to circle ABR; RAQ is a straight line; prove PQ is parallel to BR.

46.

1. ABC is a Δ; H, K are the mid-points of AB, AC; P, Q are points on BC such that $BP=\frac{1}{4}BC=\frac{1}{3}BQ$; prove $PH=QK$.

2. Find the remaining angles in Fig. 258.

3. ABCD is a parallelogram; the circle through A, B, C cuts CD at P; prove $AP=AD$.

4. APB, AQB are two circles; AP is a tangent to circle AQB; PBQ is a straight line; prove that AQ is parallel to the tangent at P.

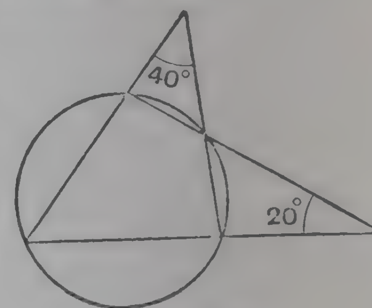


FIG. 258.

47.

1. ABCD is a square; P is a point on AB such that $AP=\frac{1}{3}AB$; Q is a point on PC such that $PQ=\frac{1}{3}PC$; prove $APQD=\frac{1}{3}ABCD$.

2. AOB is a diameter of a circle perpendicular to a chord POQ; $AO=h$, $PQ=a$; find AB in terms of a, h.

3. The side AB of a cyclic quadrilateral ABCD is produced to E; $\angle DBE=140°$, $\angle ADC=100°$, $\angle ACB=45°$; find $\angle BAC$, $\angle CAD$.

4. In ΔABC, $\angle BAC=90°$; the circle on AB as diameter cuts BC at D; the tangent at D cuts AC at P; prove $PD=PC$.

48.

1. In quadrilateral ABCD, $AB=7''$, $CD=11''$, $\angle BAD=\angle ADC=90°$, $\angle BCD=60°$; calculate AC.

2. Two chords AB, DC of a circle, centre O, are produced to meet at E; $\angle CBE=75°$, $\angle CEB=22°$, $\angle AOD=144°$; prove $\angle AOB=\angle BAC$.

3. In Fig. 259, O is the centre and TQ bisects $\angle OTP$; prove $\angle TQP=45°$.

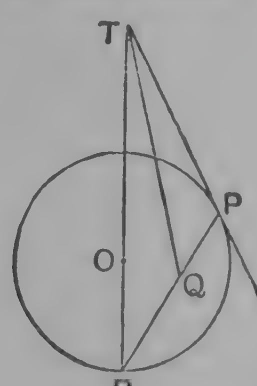


FIG. 259.

4. PAB, PBC, PCA are three unequal circles; from any point D on the circle PBC, lines DB, DC are drawn and produced to meet the circles PBA, PCA again at X, Y; prove XAY is a straight line.

49.

1. In $\triangle ABC$, $\angle ACB = 90°$, $AC = 2CB$; CD is an altitude; prove by using the figure of Pythagoras' theorem or otherwise that $AD = 4DB$.

2. In Fig. 260, O is the centre of the circle; PQ and PT are equally inclined to TO; prove $\angle QOT = 3\angle POT$.

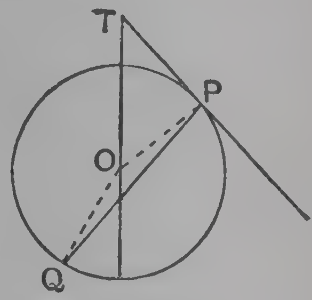


FIG. 260.

3. AOB is a chord of a circle ABC; T is a point on the tangent at A; the tangent at B meets TO produced at P; $\angle ATO = 35°$, $\angle BOT = 115°$; find $\angle BPT$.

4. In $\triangle ABC$, $AB = AC$; the circle on AB as diameter cuts BC at P; prove $BP = PC$.

50.

1. X, Y, Z are any points on the sides BC, CA, AB of the triangle ABC; prove that $AX + BY + CZ > \frac{1}{2}(BC + CA + AB)$.

2. A, B, C, D are the first milestones on four straight roads running from a town X; A is due north of D and north-west of B. C is E. 20° S. of D; find the bearing of B from C.

3. ABCD is a quadrilateral inscribed in a circle, centre O; if AC bisects $\angle BAD$, prove that OC is perpendicular to BD.

4. A diameter AB of a circle APB is produced to any point T; TP is a tangent; prove $\angle BTP + 2\angle BPT = 90°$.

51.

1. ABCD is a rectangle; P is any point on CD; prove that quad. $ABCP - \triangle APD = AD \cdot CP$.

2. ABCD is a circle; if arc $ABC = \frac{1}{4}$ arc ADC, find $\angle ADC$.

3. A, B, C are points on a circle, centre O; BO, CO are produced to meet AC, AB at P, Q; prove $\angle BPC + \angle BQC = 3\angle BAC$.

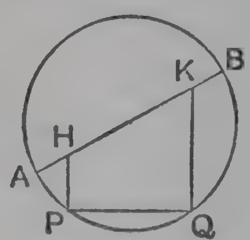


FIG. 261.

4. In Fig. 261, AB is a diameter; $\angle HPQ = \angle KQP = 90°$; prove $AH = BK$.

52

1. In $\triangle$ABC, $\angle$BAC=90°; AD is an altitude; prove that $\frac{1}{AD^2}=\frac{1}{AB^2}+\frac{1}{AC^2}$.

2. ABCD is a square inscribed in a circle; P is any point on the minor arc AB; prove $\angle$APB=3$\angle$BPC.

3. ABC is a triangle inscribed in a circle; the bisector of $\angle$BAC meets the circle at P; I is a point on PA between P and A such that PI=PB; prove $\angle$IBA=$\angle$IBC.

4. Two circles, centres A, B, cut at X, Y; XP, XQ are the tangents at X; prove $\angle$AXB is equal or supplementary to $\angle$PXQ.

53.

1. ABCD is a parallelogram; P is any point on CD; PA, PB, CB, AD cut any line parallel to AB at X, Y, Z, W; prove DCZW =2$\triangle$APY.

2. In Fig. 262, O is the centre, PQ=AO, $\angle$AOQ=90°; prove arc BR=3 arc AP.

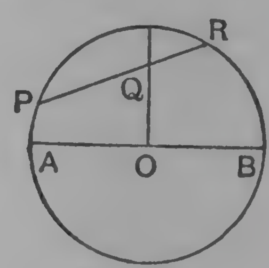


FIG. 262.

3. A rectangular strip of cardboard is 7 inches wide, 4 feet long, how many circular discs each of radius 2 inches can be cut out of it ?

4. AB, CD are parallel chords of a circle ABDC, centre O; prove $\angle$AOC equals angle between AD and BC.

54.

1. Two metre rules AOB, COD cross one another at right angles: the zero graduations are at A, C; a straight edge XY, half a metre long, moves with one end X on OB and the other end Y on OD; when the readings for X are 50, 40 cm., those for Y are 50, 60 cm. respectively. Find the readings at O.

2. Two circles PARB, QASB intersect at A, B; a line PQRS cuts one at P, R and the other at Q, S; prove $\angle$PAQ=$\angle$RBS.

3. In $\triangle ABC$, $\angle BAC = 90°$; D is the mid-point of BC; a circle touches BC at D, passes through A and cuts AC again at E; prove arc AD = 2 arc DE.

4. Two circular cylinders of radii 2″, 6″ are bound tightly together with their axes parallel by an elastic band. Find its stretched length.

55.

1. In Fig. 263, BC is an arc of radius 8″ whose centre lies on OB produced; OB = 9″, $\angle AOB = 90°$; calculate the radius of a circle touching AO, OB, and arc BC.

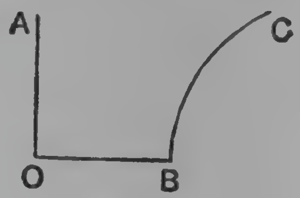


FIG. 263.

2. ABCD is a parallelogram; AB, CB are produced to X, Y; P is any point within the angle XBY; prove $\triangle PCD - \triangle PAB = \triangle ABC$.

3. $A_1 A_2 A_3 \ldots A_{20}$ is a regular polygon of 20 sides, prove that $A_1 A_8$ is perpendicular to $A_3 A_{16}$.

4. A, B, C are three points on a circle; the tangent at A meets BC produced at D; prove that the bisectors of $\angle$s BAC, BDA are at right angles.

56.

1. In $\triangle ABC$, $\angle ABC = 90°$, $\angle BAC = 45°$; the bisector of $\angle ACB$ meets AB at P; prove $AP^2 = 2PB^2$.

2. The diameter AB of a circle is produced to any point P; a line is drawn from P touching the circle at Q and cutting the tangent at A in R; prove $\angle BQP = \frac{1}{2} \angle ARP$.

3. In $\triangle ABC$, AB = AC and $\angle BAC$ is obtuse; a circle is drawn touching AC at A, passing through B and cutting BC again at P; prove arc AB = 2 arc AP.

4. The volume of a circular cylinder is V cub. in. and the area of its curved surface is S sq. in.; find its radius in terms of V, S.

SECTION V.

AREAS OF RECTANGLES.

Definition.

A rectangle contained by two lines is a rectangle whose length is equal to one of the lines and whose breadth is equal to the other.

If AB and CD are two straight lines, the area of the rectangle contained by them is equal to the product of the measures of AB and CD and is written AB . CD.

The areas of rectangles can be used to illustrate a group of arithmetic identities.

(i) $k(a+b+c+d+e) = ka + kb + kc + kd + ke.$

FIG. 264.

Draw a line and cut off from it parts PQ, QR, RS, ST, TU of lengths a, b, c, d, e inches.

Construct the rectangle PUNF so that the breadth PF is k inches: draw QG. RH, SL, TM parallel to PF to cut FN at G, H, L, M.

Then the area of PUNF equals $k(a+b+c+d+e)$ sq. inches, and the areas of the separate rectangles PG, QH, RL, SM, TN are ka, kb, kc, kd, ke sq. inches.

$$\therefore\ k(a+b+c+d+e)=ka+kb+kc+kd+ke.$$

Note.—This method can obviously be used, however many terms there are in the bracket.

(ii) $(a+b)^2=a^2+2ab+b^2$.

Draw a line PQ of length $a+b$ inches and take a point R on it such that RQ is of length b inches.

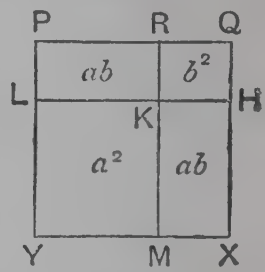


FIG. 265.

On PQ and RQ describe squares PQXY, RQHK on the same side of PQ and produce RK, HK to meet XY, PY at M, L.

Then the area of PQXY is $(a+b)^2$ sq. inches.

The areas of LKMY and RQHK are a^2 sq. inches and b^2 sq. inches.

The area of each of the rectangles PK, KX is ab sq. inches.

$$\therefore\ (a+b)^2=a^2+2ab+b^2.$$

(iii) $(a+b)(a-b)=a^2-b^2$.

Draw a line PQ of length a inches ($a>b$) and cut off a part PR of length b inches.

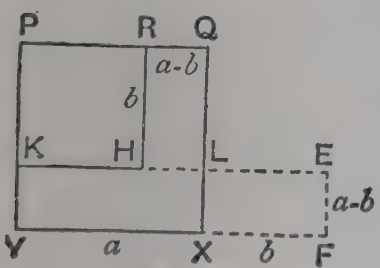


FIG. 266.

On PQ and PR describe squares PQXY, PRHK; produce KH to meet QX at L.

Produce KL, YX to E, F so that LE $=$ XF $=b$ inches.

Now

LX $=$ QX $-$ QL $=$ QX $-$ RH $=a-b$ inches.

$\therefore$ the rectangle LXFE equals the rectangle HLQR.

$\therefore$ the rectangle KYFE equals the sum of the rectangles KYXL and HLQR, and this equals PQXY $-$ PRHK $=a^2-b^2$ sq. in.

But KY $=a-b$ inches, YF $=a+b$ inches.

$$\therefore\ (a+b)(a-b)=a^2-b^2.$$

EXERCISE XLVIII.

1. Illustrate by a figure $(a+b)(c+d) \equiv ac+ad+bc+bd$.

2. Illustrate by a figure $(2a)^2 \equiv 4a^2$.

3. Illustrate by a figure $(a+b+c)^2 \equiv a^2+b^2+c^2+2bc+2ca+2ab$.

4. Illustrate by a figure $k(a-b) = ka-kb$.

5. Explain how Fig. 267 may be used to illustrate the identity $(a-b)^2 \equiv a^2-2ab+b^2$.

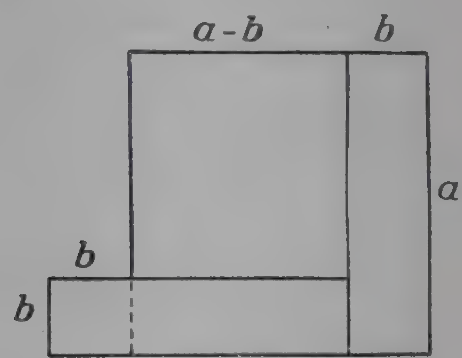


FIG. 267.

6. A straight line is divided into any two parts. What can you say about the sum of the areas of the squares on the two parts plus twice the area of the rectangle contained by the two parts ?

7. AD is an altitude of the triangle ABC, prove that the rectangle contained by the sum and difference of AB and AC is equal to the difference of the squares on BD and DC.

8. Express in words the geometrical theorem corresponding to the identity $(2a)^2 \equiv 4a^2$ or $x^2 = 4(\frac{1}{2}x)^2$.

Relations between Segments of a Straight Line.

The next theorem is given as an illustration of the application of algebra to geometry. It does not appear in any schedule of theorems required for public examinations.

THEOREM 46.

(1) If A, B, C, D are four points in order on a straight line, then $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

(2) If a straight line AB is bisected at O, and if P is any other point on AB, then $AP^2 + PB^2 = 2AO^2 + 2PO^2$.

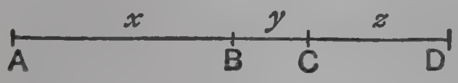


FIG. 268.

(1) Let $AB = x$ units, $BC = y$ units, $CD = z$ units.

Then $AC = x + y$, $BD = y + z$.

$$\therefore AC \cdot BD = (x+y)(y+z)$$
$$= xy + y^2 + xz + yz.$$

Also $AD = x + y + z$.

$$\therefore AB \cdot CD + AD \cdot BC = xz + (x+y+z)y$$
$$= xz + xy + y^2 + yz.$$
$$\therefore AC \cdot BD = AB \cdot CD + AD \cdot BC.$$

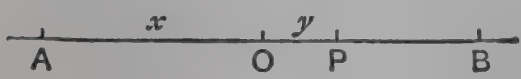


FIG. 269.

(2) Let $AO = x$ units, $OP = y$ units.

$$\therefore OB = AO = x.$$

Also $PB = OB - OP = x - y$,

and $AP = AO + OP = x + y$.

$$\therefore AP^2 + PB^2 = (x+y)^2 + (x-y)^2$$
$$= x^2 + 2xy + y^2 + x^2 - 2xy + y^2$$
$$= 2x^2 + 2y^2$$
$$= 2AO^2 + 2OP^2.$$

Q.E.D.

Note that Theorem 46 (2) is a special case of Theorem 49.

EXERCISE XLIX.

1. A straight line AB is bisected at O; P is any point on AO; prove $PO=\frac{1}{2}(PB-PA)$.

2. A straight line AB is bisected at O and produced to P; prove that $OP=\frac{1}{2}(AP+BP)$.

3. A straight line AB is bisected at O and produced to P; prove that $PA^2+PB^2=2PO^2+2OA^2$.

4. ABCD is a straight line; X, Y are the mid-points of AB, CD; prove that $AD+BC=2XY$.

5. AB is bisected at O and produced to P; prove that $AO \cdot AP = OB \cdot BP+2AO^2$.

6. AD is trisected at B, C; prove that $AD^2=AB^2+2BD^2$.

7. APB is a straight line; prove that $AB^2+AP^2=2AB \cdot AP+PB^2$.

8. AB is bisected at C and produced to P; prove that $AP^2 =4AC \cdot CP+BP^2$.

9. ABCD is a straight line; if AB=CD, prove that $AD^2+BC^2 =2AB^2+2BD^2$.

10. X is a point on AB such that $AB \cdot BX=AX^2$; prove that $AB^2+BX^2=3AX^2$.

11. C is a point on AB such that $AB \cdot BC=AC^2$; prove that $AC \cdot BC=AC^2-BC^2$.

12. X is a point on AB such that $AB \cdot BX=AX^2$; O is the mid-point of AX; prove that $OB^2=5 \cdot OA^2$.

13. AB is bisected at O and produced to P so that $OB \cdot OP=BP^2$; prove that $PA^2=5PB^2$.

14. AB is produced to P so that $PA^2=4PB^2+AB^2$; prove that $\frac{PA}{PB}=\frac{5}{2}$.

EXTENSIONS OF PYTHAGORAS' THEOREM.

Definition.

If AB and CD are any two straight lines, and if AH, BK are the perpendiculars from A, B to CD, then HK is called the *projection* of AB on CD.

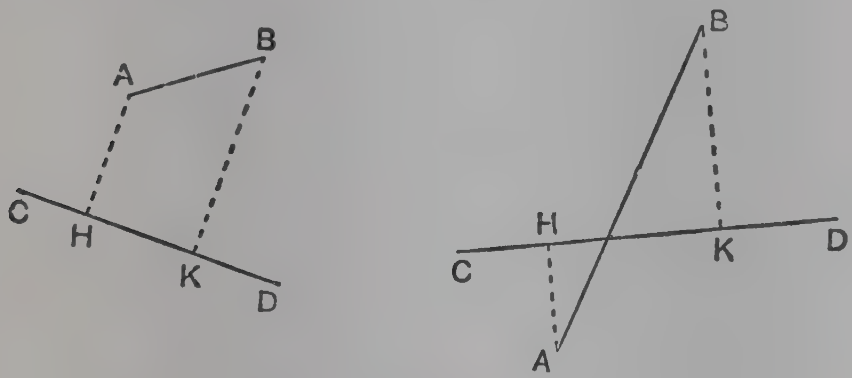


FIG. 270.

Thus, in Fig. 163, p. .

QY is the projection of BA on QP,
XC is the projection of AC on BC,
BX is the projection of QA on BC.

Or, in Fig. 271,

AN is the projection of AC on AB,
BN is the projection of BC on AB.

Figure 274, p. 211, may be used for further illustrations of the use and meaning of the word "projection."

THEOREM 47.

In an obtuse-angled triangle, the square on the side opposite the *obtuse* angle is equal to the sum of the squares on the sides containing it *plus* twice the rectangle contained by one of those sides and the projection on it of the other.

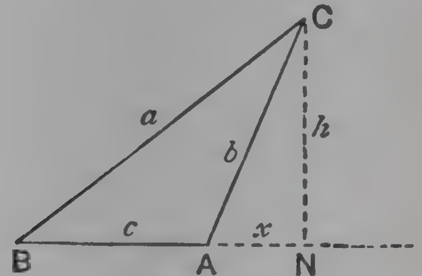


FIG. 271.

Given $\angle BAC$ is obtuse and CN is the perpendicular from C to BA produced.

To prove $BC^2 = BA^2 + AC^2 + 2BA \,.\, AN$.

(Put in a small letter for each length that comes in the answer and also for the altitude.)

Let $BC = a$ units, $BA = c$ units, $AC = b$ units, $AN = x$ units, $CN = h$ units.

It is required to prove that $a^2 = c^2 + b^2 + 2cx$.

Since $\angle BNC = 90°$, $a^2 = (c + x)^2 + h^2$,

$$\therefore\ a^2 = c^2 + 2cx + x^2 + h^2.$$

Since $\angle ANC = 90°$, $b^2 = x^2 + h^2$,

$$\therefore\ a^2 = c^2 + 2cx + b^2,$$

$$\text{or } BC^2 = BA^2 + AC^2 + 2BA \,.\, AN.$$

Q.E.D.

THEOREM 48.

In any triangle, the square on the side opposite an *acute* angle is equal to the sum of the squares on the sides containing it *minus* twice the rectangle contained by one of those sides and the projection on it of the other.

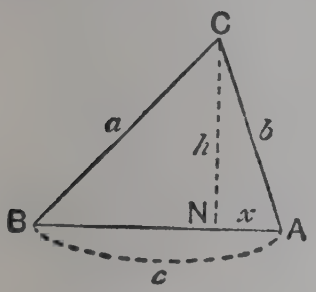


FIG. 272 (i)

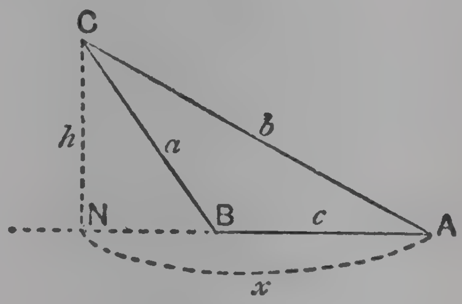


FIG. 272 (ii)

Given $\angle BAC$ is acute and CN is the perpendicular from C to AB or AB produced.

To prove $BC^2 = BA^2 + AC^2 - 2AB \,.\, AN$.

(Put in a small letter for each length that comes in the answer and also for the height.)

Let $BC = a$ units, $BA = c$ units, $AC = b$ units, $AN = x$ units, $CN = h$ units.

It is required to prove that $a^2 = c^2 + b^2 - 2cx$.

In Fig. 272 (i), $BN = c - x$; in Fig. 272 (ii), $BN = x - c$.

Since $\angle CNB = 90°$, $a^2 = (c-x)^2 + h^2$ in Fig. 272 (i),

or $a^2 = (x-c)^2 + h^2$ in Fig. 272 (ii);

$\therefore$ in each case, $a^2 = c^2 - 2cx + x^2 + h^2$.

Since $\angle ANC = 90°$, $b^2 = x^2 + h^2$,

$$\therefore\ a^2 = c^2 - 2cx + b^2,$$

$$\text{or } BC^2 = BA^2 + AC^2 - 2AB \,.\, AN.$$

Q.E.D.

Definition.

The line joining a vertex of a triangle to the mid-point of the opposite side is called a **median.**

THEOREM 49. (Apollonius' Theorem.)

In any triangle, the sum of the squares on two sides is equal to twice the square on half the base *plus* twice the square on the median which bisects the base.

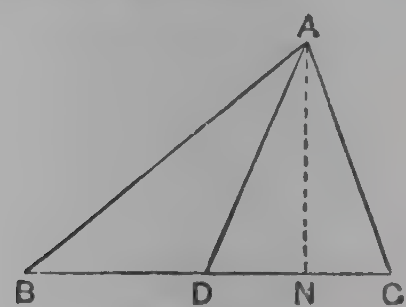


FIG. 273.

Given D is the mid-point of BC.

To prove $AB^2 + AC^2 = 2AD^2 + 2BD^2$.

Draw AN perpendicular to BC.

Of the angles ADB, ADC, suppose $\angle ADC$ is acute, so that $\angle ADB$ is obtuse.

From the triangle ADB, since $\angle ADB$ is obtuse,

$$AB^2 = AD^2 + DB^2 + 2BD \,.\, DN.$$

From the triangle ADC, since $\angle ADC$ is acute,

$$AC^2 = AD^2 + DC^2 - 2DC \,.\, DN.$$

But $BD = DC$, given;

$$\therefore\ BD \,.\, DN = DC \,.\, DN \text{ and } BD^2 = DC^2.$$

$\therefore$ adding, $\qquad AB^2 + AC^2 = 2AD^2 + 2DB^2.$

Q.E.D.

Theorems 47, 48, can also be proved by the method used for Pythagoras' theorem, and it is instructive to use this as an illustration.

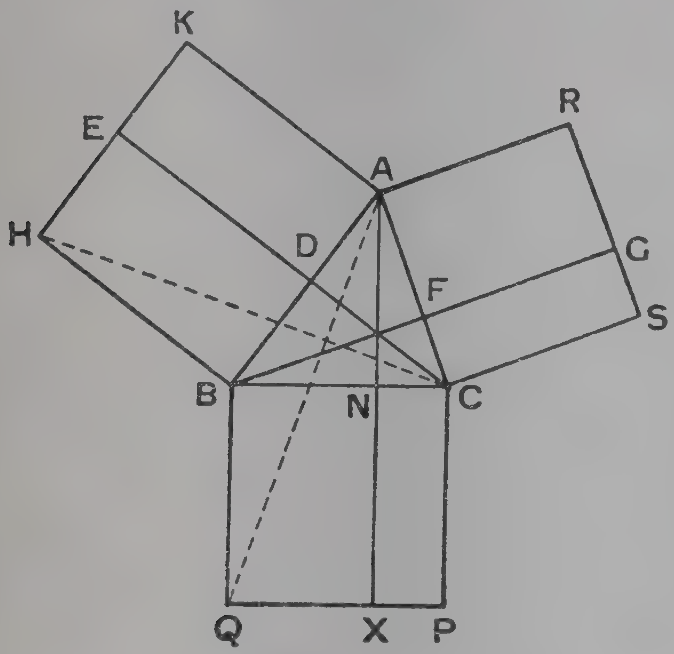


FIG. 274.

The outline of the argument required for Theorem 48 (see Fig. 274) is given below and forms a good series of riders.

(i) $\triangle ABQ \equiv \triangle HBC$.

(ii) rectangle BNXQ = rectangle BDEH ;

(iii) rectangle NCPX = rectangle CFGS ;

(iv) $\triangle KAC \equiv \triangle BAR$;

(v) rectangle ADEK = rectangle AFGR ;

(vi) $BC^2 = BA^2 + AC^2 - 2BA \,.\, AD$;

and $BC^2 = BA^2 + AC^2 - 2CA \,.\, AF$.

The proof for Theorem 47 may now be taken by a precisely similar argument, but the figure looks more complicated.

EXERCISE L.

1. Find by calculation which of the following triangles are obtuse-angled, their sides being as follows: (i) 4, 5, 7; (ii) 7, 8, 11; (iii) 8, 9, 12; (iv) 15, 16, 22.

2. Each of the sides of an acute-angled triangle is an exact number of inches; two of them are 12″, 15″; what is the greatest length of the third side?

3. In $\triangle ABC$, $BC=6$, $CA=3$, $AB=4$; CN is an altitude; calculate AN and CN.

4. In $\triangle ABC$, $BC=8$, $CA=9$, $AB=10$; CN is an altitude; calculate AN and CN.

5. In $\triangle ABC$, $BC=7$, $CA=13$, $AB=10$; CN is an altitude; calculate AN, BN, CN.

6. Find the area of the triangle whose sides are 9″, 10″, 11″.

7. ABCD is a parallelogram; $AB=5''$, $AD=3''$; the projection of AC on AB is 6″; calculate AC.

8. In $\triangle ABC$, $AC=8$ cm., $BC=6$ cm., $\angle ACB=120°$; calculate AB.

9. In $\triangle ABC$, $AB=8$ cm., $AC=7$, $BC=3$; prove $\angle ABC=60°$.

10. The sides of a triangle are 23, 27, 36; is it obtuse-angled?

11. In $\triangle ABC$, $AB=9''$, $AC=11''$, $\angle BAC > 90°$; prove $BC > 14''$.

12. In $\triangle ABC$, $AB=14''$, $BC=10''$, $CA=6''$; prove $\angle ACB=120°$.

13. The sides of a $\triangle$ are 4, 7, 9; calculate the length of the shortest median.

14. Find the lengths of the medians of a triangle whose sides are 6, 8, 9 cms.

15. The sides of a parallelogram are 5 cm., 7 cm., and one diagonal is 8 cm.; find the length of the other.

16. AD is a median of the $\triangle ABC$, $AB=6$, $AC=8$, $AD=5$; calculate BC.

17. In $\triangle ABC$, $AB=4$, $BC=5$, $CA=8$; BC is produced to D so that $CD=5$; calculate AD.

---

18. ABC is an equilateral triangle; BC is produced to D so that $BC=CD$; prove $AD^2=3AB^2$.

19. In $\triangle ABC$, $AB=AC$; CD is an altitude; prove that $BC^2=2AB\,.\,BD$.

20. BE, CF are altitudes of the triangle ABC; prove that $AF\,.\,AB=AE\,.\,AC$.

21. ABCD is a parallelogram; prove that $AC^2+BD^2=2AB^2+2BC^2$.

22. ABCD is a rectangle; P is any point in the same or any other plane; prove that $PA^2+PC^2=PB^2+PD^2$.

23. In $\triangle ABC$, $AB=AC$; AB is produced to D so that $AB=BD$; prove $CD^2=AB^2+2BC^2$.

24. In $\triangle ABC$, $\angle ACB=90°$; AB is trisected at P, Q; prove that $PC^2+CQ^2+QP^2=\frac{2}{3}AB^2$.

25. The base BC of $\triangle ABC$ is trisected at X, Y; prove that $AX^2+AY^2+4XY^2=AB^2+AC^2$.

26. AD, BE, CF are the medians of $\triangle ABC$; prove that $4(AD^2+BE^2+CF^2)=3(AB^2+BC^2+CA^2)$.

27. ABCD is a quadrilateral; X, Y are the mid-points of AC, BD; prove that $AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4XY^2$.

28. ABC is a triangle; ABPQ, ACXY are squares outside ABC; prove that $BC^2+QY^2=AP^2+AX^2$.

29. ABCD is a tetrahedron; $\angle BAC=\angle CAD=\angle DAB=90°$; prove that BCD is an acute-angled triangle.

Segments of a Chord.

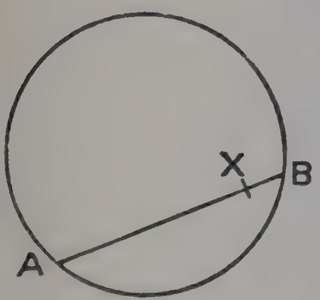


FIG. 275.

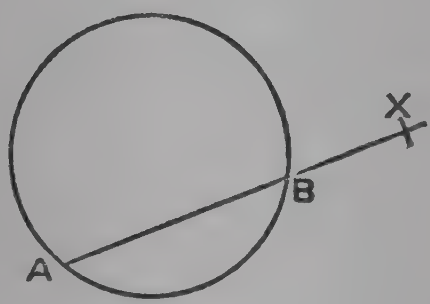


FIG. 276.

If AB is any chord of a circle, and if X is any point either on AB (Fig. 275), or AB produced (Fig. 276), then AX and BX are called the **segments of the chord** formed by the point of division X.

The rectangle contained by the segments, AX and BX, of the chord is a rectangle whose length is AX and breadth BX, so that its area is measured by AX . BX.

THEOREM 50.

If two chords of a circle intersect at a point within the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

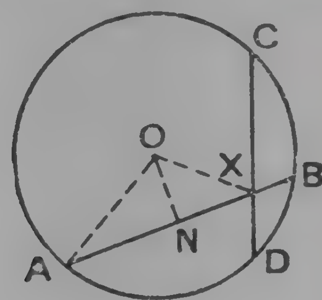


FIG. 277.

Given two chords AB, CD of a circle, intersecting at X within the circle.

To prove AX . XB = CX . XD.

Let O be the centre of the circle, draw the perpendicular ON from O to AB; join OA, OX.

Since ON is perpendicular to AB, AN = NB.

$$\begin{aligned} \therefore\ AX\,.\,XB &= (AN+NX)(NB-NX) \\ &= (AN+NX)(AN-NX) \\ &= AN^2-NX^2 \\ &= (AN^2+NO^2)-(NX^2+NO^2) \\ &= OA^2-OX^2,\ \text{Pythagoras.} \end{aligned}$$

Similarly $\quad CX\,.\,XD = OC^2 - OX^2.$

But OA = OC, radii.

$$\therefore\ AX\,.\,XB = CX\,.\,XD.$$

Q.E.D.

Corollary. *If* X *is any point inside a circle with centre* O *and radius r, then the rectangle contained by the segments of any chord drawn through* X *equals* $r^2 - OX^2$.

THEOREM 51.

If from a point without a circle a secant and a tangent to the circle are drawn, the rectangle contained by the whole secant and the segment of it without the circle is equal to the square on the tangent.

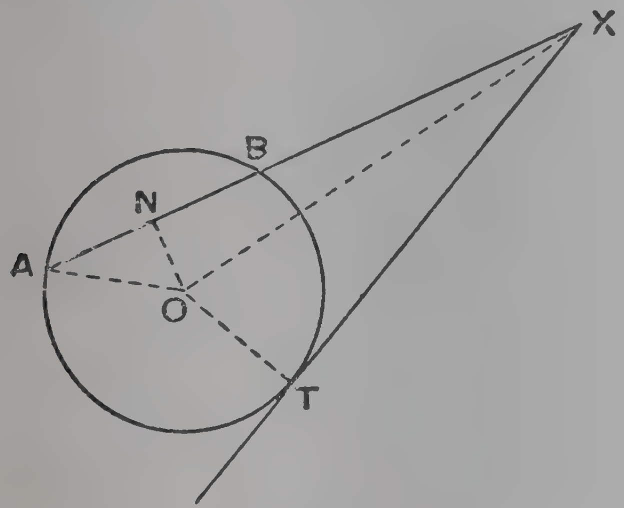


FIG. 278.

Given a secant XBA and a tangent XT touching the circle at T.

To prove $XA \cdot XB = XT^2$.

Let O be the centre of the circle, draw the perpendicular ON from O to AB; join OA, OX, OT.

Since ON is perpendicular to AB, AN = NB.

$$\therefore \ XA \cdot XB = (XN + NA)(XN - NB)$$
$$= (XN + NA)(XN - NA) = XN^2 - NA^2$$
$$= (XN^2 + NO^2) - (NA^2 + NO^2) = XO^2 - AO^2, \text{ Pythagoras,}$$
$$= XO^2 - OT^2, \ OA = OT \text{ radii,}$$
$$= XT^2, \text{ Pythagoras, since } \angle OTX = 90^\circ. \qquad \text{Q.E.D.}$$

Corollary 1. *If two chords of a circle meet when produced at a point outside the circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.*

Each is equal to the square on the tangent from that point to the circle.

Corollary 2. *If* X *is any point outside a circle with centre* O *and radius r, the rectangle contained by the segments of any chord drawn through* X *equals* $OX^2 - r^2$.

It should be noted that Theorem 50 and Corollary 1 of Theorem 51 are in reality two cases of the same theorem.

The converses of Theorems 50, 51 are important and may easily be proved by a *reductio ad absurdum* method. We may state them as follows:

I. If two straight lines AB and CD cut each other either both

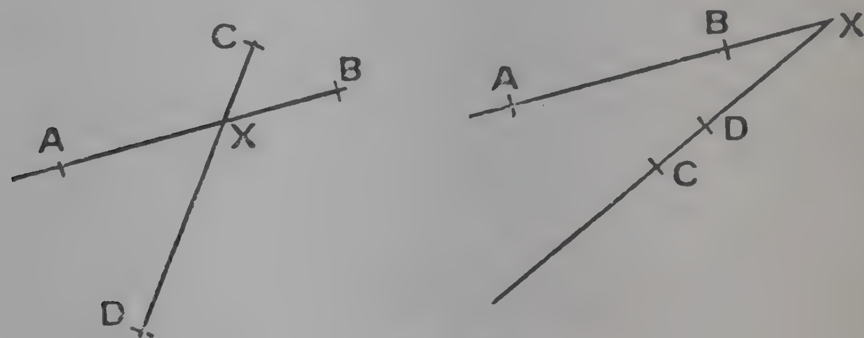


FIG. 279.

internally or both externally so that $AX \cdot XB = CX \cdot XD$, then the four points A, B, C, D lie on a circle.

II. If from any point X on a line AB produced another straight line is drawn and a point T taken on it so that

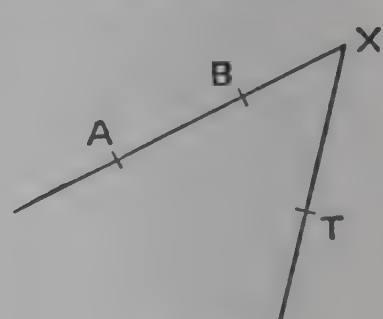


FIG. 280.

$XA \cdot XB = XT^2$ then the line XT is a tangent to the circle which passes through A, B, T.

There are two other important results which it is convenient to refer to at this stage: they are given later as corollaries to Theorem 58, page 248.

If AB is any diameter of a circle and if PN is the perpendicular to AB from any point P on the circumference, then

(i) $PN^2 = AN \,.\, NB$;

(ii) $AP^2 = AN \,.\, AB$ and $BP^2 = BN \,.\, BA$.

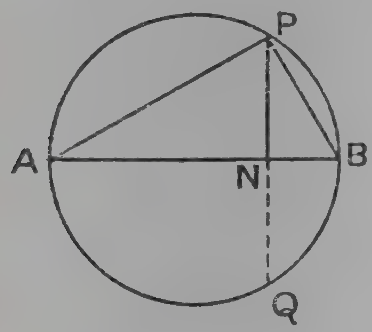


FIG. 281.

(i) Produce PN to meet the circle again at Q.

Since PQ is perpendicular to the diameter AB, $PN = NQ$;

$$\therefore\ AN \,.\, NB = PN \,.\, NQ = PN \,.\, PN = PN^2.$$

(ii) $$AP^2 = AN^2 + PN^2, \text{ Pythagoras,}$$
$$= AN^2 + AN \,.\, NB$$
$$= AN(AN + NB) = AN \,.\, AB.$$

Similarly $BP^2 = BN \,.\, BA$.

It is instructive to notice that the last result may also be proved as follows:

Either, The circle on PB as diameter passes through N.

But $\angle APB = 90°$, $\therefore$ AP is a tangent to the circle PNB;

$$\therefore\ AP^2 = AN \,.\, AB.$$

Or, From the figure and proof of Pythagoras' theorem it is at once evident that the square on AP is equal to the rectangle whose area is AN . AB.

EXERCISE LI.

1. In Fig. 277, (i) if AX=6, XB=2, CX=3, find XD.
(ii) if AB=11, BX=3, CX=4, find CD;
(iii) if AX=12, AB=15, CX=XD, find CD.

2. In Fig. 277, (i) if the radius is 7 cm. and OX=5 cm., find AX . XB;
(ii) if OA=6 cm., AX=5 cm., OX=4 cm., find BX.

3. From a point P on a circle PN is drawn perpendicular to a diameter AB; AN=4″, NB=16″; find PN.

4. In △ABC, ∠BAC=90°, AB=4″, AC=3″; AD is an altitude; find BD.

5. In △ABC, AB=9, AC=12, F is the mid-point of AC; the circle through B, F, C cuts AB at E; find BE.

6. In Fig. 282, (i) if AB=2, BX=6, DX=3; find CD;
(ii) if AB=9, BX=3, find TX;
(iii) if TX=5, DX=$2\frac{1}{2}$, find CD.

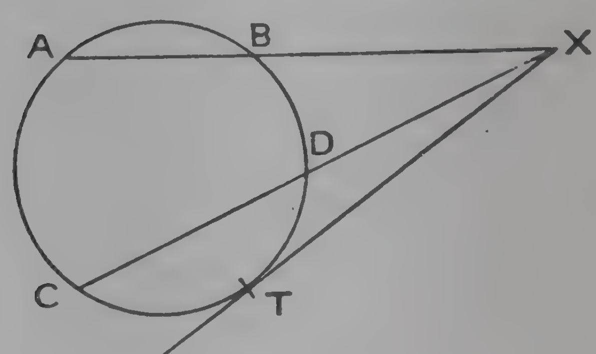


FIG. 282.

7. If in Fig. 278 the circle is of radius 5 cm. and the distance of X from the centre is 9 cm., find XA . XB.

8. ABC is a triangle inscribed in a circle; AB=AC=10 cm., BC=16 cm.; AD is drawn perpendicular to BC and is produced to meet the circle at E; find DE and the radius of the circle.

9. The roadway ACB of a bridge is a circular arc resting on supports at A, B at the same level: the highest point C of the roadway is 4 feet above AB, and AB is 8 yards. Find the radius of the arc.

10. The diagonals AC, BD of the parallelogram ABCD are of lengths 8, 10 cm. The circle through B, C, D cuts CA at F; find AF.

11. A garden roller rests against a step AB; AB=9″, AP=15″, find the diameter of the roller.

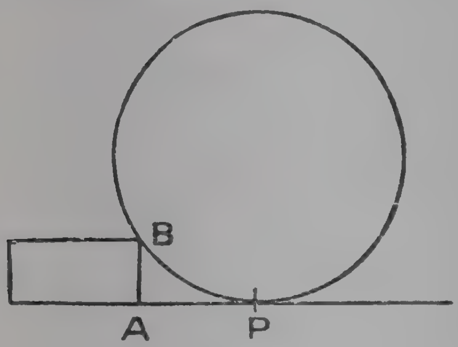


FIG. 283.

12. A small heavy body is suspended from a fixed point by a string $6\frac{1}{2}$ feet long; it is pulled aside, the string remaining taut, so that it rises 6 inches. Through what *horizontal* distance does it move?

13. In the triangle XBC, XB=6 cm., XC=4 cm., ∠BXC=90°; a circle is drawn to touch BX at B and to pass through C. Find the radius of the circle.

14. AXB, CXD are two perpendicular chords of a circle whose centre is O; AX=3″, CX=5″, XD=6″; find OX and the radius of the circle.

15. (i) If the mean radius of the Earth is r miles, and if a man stands on a hill of height h miles above mean level, show that the distance he can see is about $\sqrt{2rh}$ miles.

(ii) Taking the radius of the Earth as 3950 miles, show that at a height of x *feet* above sea level, the distance that can be seen on level ground is about $\sqrt{\frac{3x}{2}}$ *miles.*

(iii) Find the approximate distance of the horizon for a height of (i) 6 feet, (ii) 600 feet.

16. If, in Fig. 282, XC=2XT, prove that CD=3DX.

17. Two circles intersect at A, B; P is any point on AB produced; prove that the tangents from P to the two circles are equal.

18. Any two circles being given, a third circle is drawn cutting one of them at A, B and the other at C, D; the lines AB and CD are produced to meet at X; prove that the tangents from X to the two first circles are equal.

19. Prove that the common chord of two intersecting circles bisects their common tangents.

20. In $\triangle ABC$, $AB = AC$ and $\angle BAC = 36°$; the bisector of $\angle ABC$ meets AC at P; prove that $AC \cdot CP = BC^2 = AP^2$.

21. The altitudes BE, CF of $\triangle ABC$ intersect at H; prove that (i) $BH \cdot HE = CH \cdot HF$; (ii) $AF \cdot AB = AE \cdot AC$; (iii) $CE \cdot CA = CH \cdot CF$.

22. In $\triangle ABC$, $\angle BAC = 90°$, $AB = 2AC$. If AD is an altitude, prove that $BD = 4DC$.

23. PQ is a chord of a circle, centre O; the tangents at P, Q meet at X; OX cuts PQ at N; prove that $ON \cdot OX = OP^2$.

24. Two circles intersect at A, B; X is a point such that the tangents from X to the circles are equal; prove that X must lie on AB produced.

25. AB is a diameter of a circle; PQ is a chord; the tangent at B meets AP, AQ produced at X, Y; prove that $AP \cdot AX = AQ \cdot AY = AB^2$.

26. AB, AC are two chords of a circle; any line parallel to the tangent at A cuts AB, AC at D, E; prove that $AB \cdot AD = AE \cdot AC$.

27. Two lines XAB, XCD cut a circle at A, B, C, D; through X a line is drawn parallel to BC to meet DA produced at Y; prove that $XY^2 = YA \cdot YD$.

28. Three circles are drawn so that each intersects the other two; prove that the three common chords are concurrent. (If AB, CD, EF are the common chords, and if AB cuts CD at X, suppose if possible that EX when produced cuts the circle at distinct points P, Q.)

29. The triangle ABC is such that AC equals the diagonal of the square described on AB; D is the mid-point of AC; prove $\angle ABD = \angle ACB$.

30. In $\triangle ABC$, $\angle BAC = 90°$; E is a point on BC such that $AE = AB$; prove that $BE \cdot BC = 2AB^2$.

CONSTRUCTION 20.

(i) Construct a square equal in area to a given rectangle.
(ii) Construct a square equal in area to a given polygon.

(i) *Given* a rectangle ABCD.

To construct a square of equal area.

Produce AB to E, making BE = BC.

On AE as diameter, describe a semicircle.

Produce CB to meet the semicircle at P.

On BP describe a square.

This is the required square.

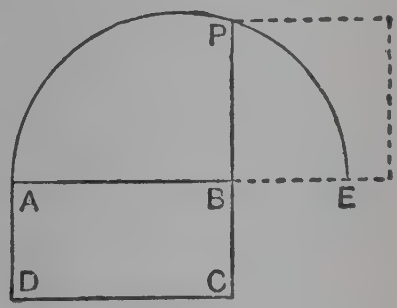


FIG. 284.

Proof. $BP^2 = AB \,.\, BE$, but $BE = BC$;

$\therefore$ $BP^2 = AB \,.\, BC =$ area of ABCD. Q.E.F.

(ii) *Given* any polygon.

To construct a square of equal area.

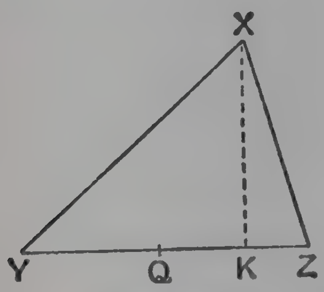


FIG. 285.

By the method of Constr. 12, reduce the polygon to any equivalent triangle XYZ (see p. 113).

Draw the altitude XK and bisect YZ at Q.

Use (i) to construct a square of area equal to a rectangle whose sides are equal to YQ and XK.

This is the square required.

Proof. Area of polygon = area of $\triangle XYZ$.

$= \frac{1}{2} YZ \,.\, XK$

$= YQ \,.\, XK =$ square. Q.E.F.

EXERCISE LII.

1. Draw a rectangle of sides 5 cm., 8 cm., and construct a square of equal area ; measure its side.

2. Construct a line of length $\sqrt{31}$ cm., and then measure it.

3. Use Construction 20 to solve the simultaneous equations $x+y=11$, $xy=24$.

4. Use Construction 20 to solve the quadratic $x^2-9x+12=0$.

5. Construct a square equal in area to an equilateral triangle of side 3 inches ; measure its side.

6. Take three points A, B, C in order on a straight line and construct a point P on the line so that $AP^2=AB.AC$. (Start by drawing the circle on AC as diameter.)

7. Given a line CD and two points A, B on the same side of CD. If a circle is drawn through A and B to touch CD, construct the two possible positions of the points of contact. (Start by joining AB and producing it to cut CD at X.)

8. Draw a regular hexagon of side 5 cm. and construct a square of equal area ; measure its side.

9. Solve by a construction the equations $x-y=4$, $xy=25$.

10. Construct a rectangle of perimeter 20 cm. and area 20 sq. cm. ; measure its sides. Show that if the perimeter of a rectangle is given its area is greatest when the figure is a square.

PART III.

SECTION VI.

RATIO AND SIMILAR FIGURES.

RATIO.

If the lengths of two straight lines are 4 cm. and 6 cm., we say that the first is $\frac{4}{6}$ or $\frac{2}{3}$ of the second, and that $\frac{2}{3}$ is the ratio of the lengths of the two lines; this is often written 2 : 3. In general, if two quantities contain respectively a units and b units *of the same kind*, we say that their ratio is $\frac{a}{b}$ or $a : b$. The ratio is therefore a comparison of their magnitudes. The quantities must of course be of the same kind, it would be meaningless to compare 5 yards with 10 shillings.

If two quantities have a common measure, we can express their ratio as the ratio of two **integers**, *e.g.* if the lengths of two lines are 2·56 in., 1·12 in., their ratio is $\frac{2 \cdot 56}{1 \cdot 12} = \frac{256}{112} = \frac{16}{7}$ or $16 : 7$. Here a common measure is $\frac{1}{100}$ inch. But we frequently meet lines whose lengths have no common measure; if the side of a square is 1 inch, the diagonal is $\sqrt{2}$ inches (Pythagoras), and these two lengths have no common measure and are called **incommensurable**. The ratio of two such lengths cannot be expressed as the ratio of two integers, although we can find two integers whose ratio differs from this ratio by an amount as small as we please. Formal proofs of theorems involving the ratios of incommensurable quantities are very difficult, and we shall assume that, if a theorem has been proved for all commensurable ratios, it also remains true if the ratios are incommensurable.

If four quantities a, b, c, d are such that the ratio of a to b equals the ratio of c to d, then a, b, c, d are said to be in proportion.

If a, b, c, d are in proportion, we have $\frac{a}{b}=\frac{c}{d}$, and d is called the fourth proportional to a, b, c.

If three quantities a, b, c are such that $\frac{a}{b}=\frac{b}{c}$, they are said to be in **continued proportion**. Further, c is called the third proportional to a, b, and b is called the **mean proportional** between a, c.

FIG. 286.

FIG. 287.

If P is any point on a straight line AB or on AB produced, AP and PB are called segments of the line, and the line AB is said to be divided at P in the ratio $\frac{AP}{PB}$.

If P lies between A and B, the line AB is said to be **divided internally** in the ratio $\frac{AP}{PB}$.

If P lies on AB produced or BA produced, the line AB is said to be **divided externally** in the ratio $\frac{AP}{PB}$.

It is important to notice that in all cases, Fig. 286 and Fig. 287, AB is the *whole* line and does *not* appear in the ratio $\frac{AP}{BP}$; the "segments" of AB are AP and PB whether P lies on AB or on AB produced. This aspect may also be emphasised, if considered advisable, by a discussion on directed lengths and the interpretation of positive and negative ratios.

EXERCISE LIII.

1. What is the value of the following ratios: (i) 3 in. : 2 ft.; (ii) $4d. : 2s.$; (iii) 20 min. : $1\frac{1}{2}$ hr.; (iv) 3 sq. ft. : 2 sq. yd.; (v) 3 right angles : 120°; (vi) 3 m. : 25 cm.?

2. Find x in the following: (i) $3 : x = 4 : 10$; (ii) x feet : 5 yards $= 2 : 3$; (iii) $6 : x = x : 24$; (iv) 2 hours : 50 minutes = 3 shillings : x shillings.

3. If $\frac{a}{b} = \frac{c}{d}$, prove that:

(i) $\frac{b}{a} = \frac{d}{c}$; (ii) $ad = bc$; (iii) $\frac{a+b}{b} = \frac{c+d}{d}$;

(iv) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$; (v) $\frac{b+d}{a+c} = \frac{b}{a}$.

4. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, fill up the blank spaces in the following:

(i) $\frac{a}{a+b} = \frac{c}{\quad}$; (ii) $\frac{a-b}{a} = \frac{c-d}{\quad}$; (iii) $\frac{a+c}{b+d} = \frac{\quad}{d+f}$;

(iv) $\frac{a}{b} = \frac{\quad}{b+d+f}$; (v) $\frac{a-3c}{b-3d} = \frac{2a+7c-23e}{\quad}$; (vi) $\frac{ac}{db} = \frac{a^2+e^2}{\quad}$.

5. Solve the equations (i) $\frac{x+\frac{1}{2}}{x-\frac{1}{2}} = \frac{7}{3}$; (ii) $\frac{5x^2-3x+2}{5x^2+3x-2} = \frac{5x-1}{5x+1}$.

6. Are the following in proportion (i) $3\frac{1}{3}$, 5, 8, 12; (ii) 8 inches, 6 degrees, 12 degrees, 9 inches?

7. Find the fourth proportional to (i) 2, 3, 4; (ii) ab, bc, cd.

8. Find the third proportional to (i) $\frac{1}{2}$, $\frac{1}{6}$; (ii) x, xy.

9. Find a mean proportional between (i) 4, 25; (ii) a^2b, bc^2.

10. A line AB, 8″ long, is divided internally at P in the ratio 2 : 3; find AP.

11. A line AB, 8″ long, is divided externally at Q in the ratio 7 : 3; find BQ.

12. AB is divided internally at C in the ratio 5 : 6. Is C nearer to A or B?

13. AB is divided externally at D in the ratio 9 : 7. Is D nearer to A or B?

14. AB is divided externally at D in the ratio 3 : 5. Is D nearer to A or B ?

15. A line AB, 6″ long, is divided internally at P in the ratio 2 : 1, and externally at Q in the ratio 5 : 2 ; find the ratios in which PQ is divided by A and B.

16. ABCDE is a straight line such that AB : BC : CD : DE =1 : 3 : 2 : 5. Find the ratios (i) $\frac{AB}{AE}$; (ii) $\frac{AC}{CE}$; (iii) $\frac{EB}{AD}$.

Find the ratios in which BE is divided by A and D.

If BE=4″, find AC.

17. A line AB, 8″ long, is divided internally at C and externally at D in the ratio 7 : 3; O is the mid-point of AB; prove that OC . OD=OB².

18. A line AB, 6″ long, is divided internally at C and externally at D in the ratio 4 : 1; O is the mid-point of CD; prove that AO=16 BO; and find the length of CD.

19. A line of length x'' is divided internally in the ratio $a : b$; find the lengths of the parts.

20. A line of length y'' is divided externally in the ratio $a : b$; find the lengths of the parts.

21. A line AB is bisected at O and divided at P in the ratio $x : y$; find the ratio $\frac{OP}{AB}$.

22. AB is divided internally at C and externally at D in the ratio $x : y$; find (i) $\frac{CD}{AB}$, (ii) the ratio in which B divides CD.

23. ABCDEF is a straight line such that AB : BC : CD : DE : EF $=p : q : r : s : t$; find (i) $\frac{AB}{AF}$, (ii) $\frac{BE}{CF}$, (iii) the ratios in which A and E divide CF. If BD=x'', find AE.

24. ABCD, AXYZ are two straight lines such that AB : BC : CD =AX : XY : YZ. Fill up the blank spaces in the following :

(i) $\frac{AB}{AX}=\frac{AC}{\quad}$; (ii) $\frac{BC}{AD}=\frac{\quad}{AZ}$; (iii) $\frac{XZ}{AY}=\frac{\quad}{AC}$.

25. ABC is a straight line; if AC=λ . AB, find $\frac{AB}{BC}$ in terms of λ.

26. The sides of a triangle are in the ratio $x : y : z$ and its perimeter is p inches; find the sides.

THEOREM 52.

A straight line drawn parallel to one side of a triangle divides the sides proportionally.

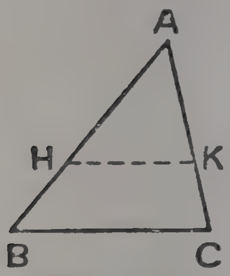


FIG. 288 (1).

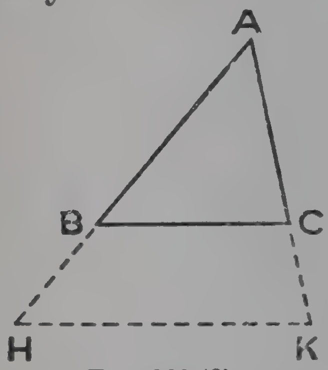


FIG. 288 (2).

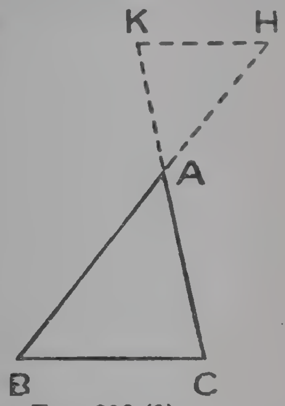


FIG. 288 (3).

Given a line parallel to BC cuts AB, AC (produced if necessary) at H, K.

To prove $$\frac{AH}{HB}=\frac{AK}{KC}.$$

Express $\frac{AH}{HB}$ as a fraction $\frac{x}{y}$, where x and y are integers.

(This assumes that AH and HB are commensurable.)

Divide AH into x equal parts and HB into y equal parts ; then each part of AH is equal to each part of HB.

Through each point of division draw a line parallel to BC.

These lines divide AK into x equal parts and KC into y equal parts, and each part of AK is equal to each part of KC.

$$\therefore \frac{AK}{KC}=\frac{x}{y}; \qquad \therefore \frac{AH}{HB}=\frac{AK}{KC}.$$ Q.E.D.

Corollary. If HK is parallel to BC, $\frac{AH}{AB}=\frac{AK}{AC}$ and $\frac{HB}{AB}=\frac{KC}{AC}$.

These may be proved in exactly the same way.

Note.—It is not possible to express $\frac{AH}{HB}$ as the ratio of two integers, unless AH and HB are commensurable.

If AH and HB are incommensurable, the theorem is still true, but the difficulty of its proof makes it unsuitable for an elementary course.

THEOREM 53.

If two sides of a triangle are divided in the same ratio, the straight line joining the points of section is parallel to the third side.

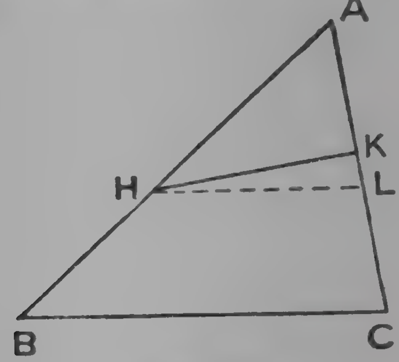


FIG. 289.

Given $$\frac{AH}{HB}=\frac{AK}{KC}.$$

To prove HK is parallel to BC.

Draw HL parallel to BC to cut AC at L.

$$\frac{AH}{HB}+1=\frac{AK}{KC}+1, \quad \therefore \frac{AH+HB}{HB}=\frac{AK+KC}{KC};$$

$$\therefore \frac{AB}{HB}=\frac{AC}{KC}.$$

But $$\frac{AB}{HB}=\frac{AC}{LC}, \text{ since HL is } \| \text{ BC};$$

$$\therefore \frac{AC}{KC}=\frac{AC}{LC};$$

$$\therefore KC=LC;$$

$\therefore$ K coincides with L and HK coincides with HL.

But HL is $\|$ BC, $\therefore$ HK is $\|$ BC. Q.E.D.

Corollary 1. If $\frac{AH}{AB}=\frac{AK}{AC}$, HK is parallel to BC.

2. If $\frac{HB}{AB}=\frac{KC}{AC}$, HK is parallel to BC.

Note.—Alternative methods of proof of Theorems 52, 53 are given in Appendix I. It is instructive to take them as riders, see Ex. 12, 13.

EXERCISE LIV.

1. In Fig. 288, AB $=10\cdot5$ cm., AC $=7$ cm., BH $=4\cdot5$; calculate KC.

2. In Fig. 288, AB $=19\cdot5$ cm., AC $=13$ cm., AH $=12$ cm.; calculate KC.

3. If in Fig. 288, AH $=4\frac{5}{6}$ in., HB $=2\frac{3}{10}$ in., what is the largest unit of length that can be chosen if AH and HB each contain an integral number of these units ?

4. If in Fig. 288 the unit of length on AB is $\frac{1}{180}$ inch, and if AH contains 640 units and HB contains 243 units, and if AC $=1$ inch, how many lines besides HK, BC are drawn in order to prove Theorem 52, and what is the length of each intercept on AC. Find also the lengths of AK, KC.

5. If in Fig. 288 (2), AB $=5$, AH $=7$, CK $=5$, calculate AC.

6. If in Fig. 288 (3), AH $=2$, AC $=2\frac{1}{2}$, BH $=5$, calculate CK.

7. ACE, BDF are two straight lines cut by three parallel lines AB, CD, EF ; AC $=2''$, CE $=3''$, BF $=4''$; calculate BD.

8. P is any point on the side AB of the triangle ABC ; PB is divided internally at Q in the ratio 2 : 3 ; PM, QN, BK are drawn perpendicular to AC to cut it at M, N, K ; if AM $=4$ cm., AK $=7$ cm., calculate AN.

9. ABC is a triangle, P is a point on BC such that BP $=6$ cm., PC $=4$ cm. Calculate the ratio $\frac{\triangle APB}{\triangle APC}$.

10. If in Fig. 288, $\frac{AH}{HB}=\frac{4}{3}$, what can you say about the ratios (i) $\frac{\triangle AHK}{\triangle BHK}$, (ii) $\frac{\triangle AHK}{\triangle CHK}$? What do you know about $\triangle$BHK and $\triangle$CHK ?

---

11. Three parallel lines AX, BY, CZ cut two lines ABC, XYZ ; prove that $\frac{AB}{BC}=\frac{XY}{YZ}$.

12. If two triangles have equal heights, prove that the ratio of their areas equals the ratio of their bases.

13. H and K are two points on the sides AB, AC respectively of the triangle ABC ; (i) prove that $\frac{\triangle AHK}{\triangle BHK}=\frac{AH}{HB}$, (ii) express $\frac{AK}{KC}$ as the ratio of the areas of two triangles, (iii) use these results to prove that if HK is parallel to BC then $\frac{AH}{HB}=\frac{AK}{KC}$, and conversely.

14. ABC is a triangle; P, Q are points on AB, AC such that $AP=\frac{1}{3}AB$ and $CQ=\frac{1}{3}CA$; prove that the line through C parallel to PQ bisects AB.

15. The diagonals of the quad. ABCD intersect at O; if AB is parallel to DC, prove $\frac{AO}{AC}=\frac{BO}{BD}$.

16. A line parallel to BC cuts AB, AC at H, K; prove that $AH \cdot AC=AK \cdot AB$.

17. O is any point inside the $\triangle ABC$; a line XY parallel to AB cuts OA, OB at X, Y; YZ is drawn parallel to BC to cut OC at Z; prove XZ is parallel to AC.

18. ABCD is a quadrilateral; P is any point on AB; lines PX, PY are drawn parallel to AC, AD to cut BC, BD at X, Y; prove XY is parallel to CD.

19. D is the foot of the perpendicular from A to the bisector of $\angle ABC$; a line from D parallel to BC cuts AC at X; prove $AX=XC$.

20. In Fig. 290, prove $\frac{\triangle ABC}{\triangle ABD}=\frac{CO}{OD}$.

21. I is the in-centre of $\triangle ABC$; prove that $\triangle IBC : \triangle ICA : \triangle IAB=BC : CA : AB$.

22. In Fig. 290, prove $\frac{\triangle ACD}{\triangle BCD}=\frac{AO}{BO}$.

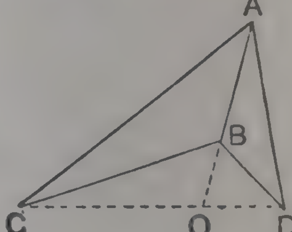


FIG. 290.

23. In Fig. 291, $AH=HB$, $AK=2KC$; find the ratio of the areas of the small triangles in the figure; hence find the ratio $\frac{CO}{OH}$.

24. ABC is a $\triangle$; H, K are points on AB, AC such that $HB=\frac{1}{4}AB$ and $KC=\frac{1}{3}AC$; BK cuts CH at O; prove $BO=OK$ and $CO=2OH$. (Use method of Ex. 23.)

25. Two circles APQ, AXY touch at A; APX, AQY are straight lines; prove $\frac{AP}{PX}=\frac{AQ}{QY}$.

26. ABC is a $\triangle$; three parallel lines AP, BQ, CR meet BC, CA, AB (produced if necessary) at P, Q, R; prove that $\frac{BP}{PC}\times\frac{CQ}{QA}\times\frac{AR}{RB}=1$.

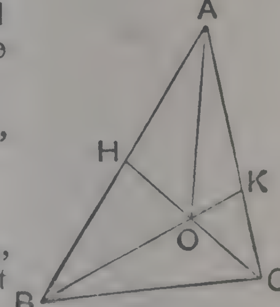


FIG. 291.

27. ABC is a triangle; a line cuts BC produced, CA, AB at P, Q, R; CX is drawn parallel to PQ, meeting AB at X; prove

$$\text{(i) } \frac{BP}{PC}=\frac{BR}{RX}; \quad \text{(ii) } \frac{BP}{PC}\times\frac{CQ}{QA}\times\frac{AR}{RB}=1.$$

(This is known as *Menelaus*' Theorem.)

CONSTRUCTION 21.

Divide a given finite straight line in a given ratio (i) internally, (ii) externally.

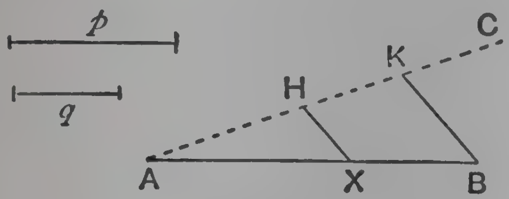


FIG. 292.

Given two lines p, q and a finite line AB.

To construct (i) a point X in AB such that $\frac{AX}{XB}=\frac{p}{q}$;

(ii) a point Y in AB produced such that $\frac{AY}{BY}=\frac{p}{q}$.

(i) Draw any line AC and cut off successively $AH=p$, $HK=q$. Join KB. Through H draw a line parallel to KB to cut AB at X.

Then $\frac{AX}{XB}=\frac{AH}{HK}=\frac{p}{q}$ by parallels. Q.E.F.

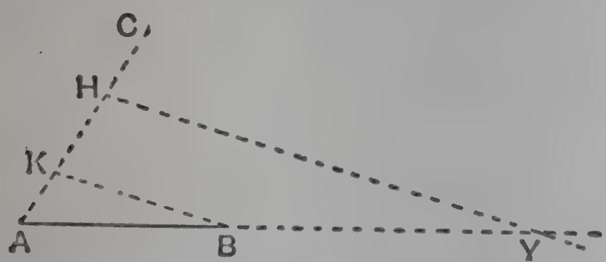


FIG. 293.

(ii) Draw any line AC; cut off $AH=p$, and from HA cut off $HK=q$. Join KB. Through H draw a line parallel to KB to cut AB produced at Y.

Then $\frac{AY}{BY}=\frac{AH}{KH}=\frac{p}{q}$ by parallels. Q.E.F.

CONSTRUCTION 22.

Construct a fourth proportional to three given lines.

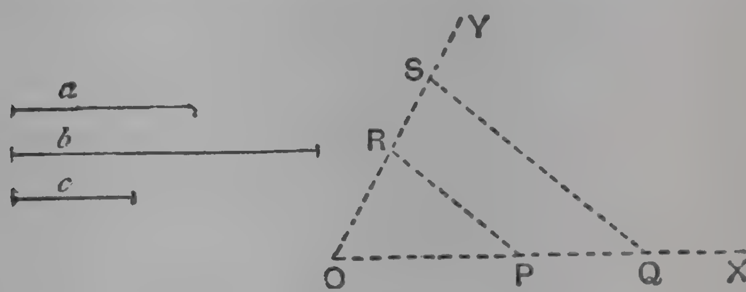


FIG. 294

Given three lines of lengths a, b, c units.

To construct a line of length d units, such that $\frac{a}{b}=\frac{c}{d}$.

Draw any two lines OX, OY.
From OX cut off parts OP, OQ such that OP $=a$, OQ $=b$.
From OY cut off a part OR such that OR $=c$.
Join PR.
Through Q draw a line QS parallel to PR to meet OY at S.
Then OS is the required fourth proportional.
Proof. Since PR is parallel to QS

$$\frac{\text{OP}}{\text{OQ}}=\frac{\text{OR}}{\text{OS}}, \qquad \therefore \frac{a}{b}=\frac{c}{\text{OS}}. \qquad \text{Q.E.F.}$$

Note.—Constructing a third proportional to two given lines. lengths a, b units, is the same as constructing a fourth proportional to three lines of length a, b, b units.

EXERCISE LV.

1. Construct and measure a fourth proportional to lines of length 4, 5, 6 cm.

2. Construct and measure a third proportional to lines of length 5, 6 cm.

3. Draw a line AB and divide it internally in the ratio 2 : 3.

4. Draw a line AB and divide it externally (i) in the ratio 5 : 3; (ii) in the ratio 3 : 5.

5. Draw a line AB and divide it internally and externally in the ratio 3 : 7.

6. Use a construction to solve $\frac{x}{3}=\frac{7}{5}$.

7. Find graphically the value of (i) $\frac{2{\cdot}3 \times 5{\cdot}9}{4{\cdot}7}$; (ii) $3{\cdot}8 \times 2{\cdot}7$.

8. Construct a line of length $\frac{11}{7}$ cms.

9. Draw a line AB and divide it in the ratio 2 : 7 : 3.

10. Draw any triangle ABC and any line PQ; construct a triangle such that its perimeter equals PQ and its sides are in the ratio AB : BC : CA.

11. Given two lines AB, AC and a point D between them, construct a line through D, cutting AB, AC at P, Q such that PD $=\frac{2}{3}$DQ.

12. Draw a line ABCD; if AB $=x$ cm., BC $=y$ cm., CD $=z$ cm., construct a line of length xyz cm.

THEOREM 54.

The bisector (internal or external) of an angle of a triangle divides the opposite side (internally or externally) in the ratio of the sides containing the angle bisected.

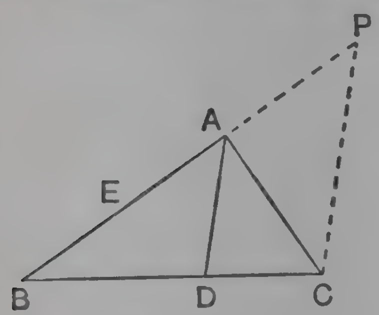


FIG. 295 (1).

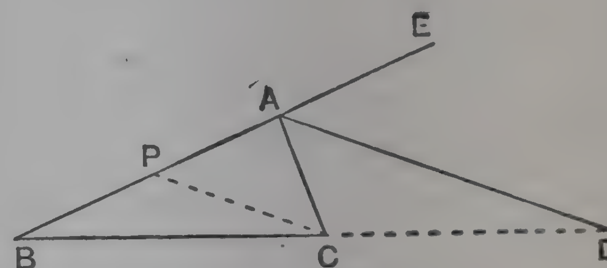


FIG. 295 (2).

Given AD bisects $\angle$BAC, internally in Fig. (1), externally in Fig. (2), and cuts BC or BC produced at D.

To prove $\dfrac{BD}{DC}=\dfrac{BA}{AC}$.

Through C draw CP parallel to DA to cut BA or BA produced at P.

Take any point E on BA in Fig. (1), and on BA produced in Fig. (2).

Then $\angle EAD = \angle APC$, corresp. $\angle$s., AD $\parallel$ PC.

$\angle DAC = \angle ACP$, alt. $\angle$s., AD $\parallel$ PC.

But $\angle EAD = \angle DAC$ given, $\therefore$ $\angle APC = \angle ACP$;

$\therefore$ AP = AC.

Since AD is parallel to PC, $\dfrac{BD}{DC}=\dfrac{BA}{AP}$.

But AP = AC, $\therefore$ $\dfrac{BD}{DC}=\dfrac{BA}{AC}$. Q.E.D.

Corollary. If the base BC is divided internally or externally at D in the ratio AB : AC, then AD bisects the angle BAC internally or externally.

This is proved by the same argument as that used to prove Theorem 54, in a reversed order.

EXERCISE LVI.

1. In $\triangle ABC$, $AB=6$ cm., $BC=5$ cm., $CA=4$ cm.; the internal and external bisectors of $\angle BAC$ cut BC and BC produced at P, Q; find BP and BQ, and show that $\frac{1}{BP}+\frac{1}{BQ}=\frac{2}{BC}$.

2. In $\triangle ABC$, $AB=4''$, $BC=3''$, $CA=5''$; the bisector of $\angle ACB$ cuts AB at D; find CD.

3. In $\triangle ABC$, $AB=12$, $BC=15$, $CA=8$; P is a point on BC such that $BP=9$; prove AP bisects $\angle BAC$; if the external bisector of $\angle BAC$ cuts BC produced at Q, and if D is the mid-point of BC, prove that $DP \,.\, DQ=DC^2$.

4. The internal and external bisectors of $\angle BAC$ meet BC and BC produced at P, Q; $BP=5$, $PC=3$; find CQ.

5. ABCD is a rectangular sheet of paper; $AB=4''$, $BC=3''$; the edge BC is folded along BD and the corner is then cut off along the crease; find the area of the remainder.

6. In $\triangle ABC$, $AB=6''$, $AC=4''$; the bisector of $\angle BAC$ meets the median BE at O; the area of $\triangle ABC$ is 8 sq. in.; what is the area of $\triangle AOB$?

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7. The internal and external bisectors of $\angle BAC$ cut BC and BC produced at P, Q; prove $\frac{BP}{PC}=\frac{BQ}{CQ}$.

8. AX is a median of $\triangle ABC$; the bisectors of $\angle$s AXB, AXC meet AB, AC at H, K; prove HK is parallel to BC.

9. ABCD is a parallelogram; the bisector of $\angle BAD$ meets BD at K; the bisector of $\angle ABC$ meets AC at L; prove LK is parallel to AB.

10. The tangent at a point A of a circle, centre O, meets a radius OB at T; D is the foot of the perpendicular from A to OB; prove $\frac{DB}{BT}=\frac{AD}{AT}$.

11. The bisector of $\angle BAC$ cuts BC at D; circles with B, C as centres are drawn through D and cut BA, CA at H, K; prove HK is parallel to BC.

12. H is any point inside the $\triangle ABC$; the bisectors of $\angle$s BHC, CHA, AHB cut BC, CA, AB at X, Y, Z; prove $\frac{BX}{XC}\times\frac{CY}{YA}\times\frac{AZ}{ZB}=1$.

13. Two circles, centres A, B, touch at O; any line parallel to AB cuts the circles at P, Q respectively; AP and BQ are produced to meet at K; prove OK bisects $\angle AKB$.

14. A straight line cuts four lines OP, OQ, OR, OS at P, Q, R, S; if $\angle POR=90°$ and OR bisects $\angle QOS$, prove $\frac{PQ}{PS}=\frac{QR}{RS}$.

15. In $\triangle ABC$, $\angle BAC = 90°$ and AD is an altitude; the bisector of $\angle ABC$ meets AD, AC at L, K; prove $\frac{AL}{LD} = \frac{CK}{KA}$.

16. ABCD is a quadrilateral; if the bisectors of $\angle$s DAB, DCB meet on DB, prove that the bisectors of $\angle$s ABC, ADC meet on AC.

17. Two circles touch internally at O; a chord PQ of the larger touches the smaller at R; prove $\frac{OP}{OQ} = \frac{PR}{RQ}$.

18. The internal and external bisectors of $\angle APB$ meet AB at X, Y; prove $\angle XPY = 90°$. If A, B are fixed points and if P varies so that $\frac{PA}{PB}$ is constant, prove that the locus of P is a circle. (*Apollonius*' circle.)

Congruence and Similarity.

Two figures of the same shape are called **similar figures.** When we make a drawing to scale of a given figure, we are constructing a figure of the same shape as the given figure, but of any convenient size. The scale drawing and the original given figure are similar figures. A lantern slide and the picture projected on the screen from the slide is a familiar example of similar figures (neglecting any distortion). The map of a small piece of flat country is an attempt to represent on a small scale the features of the country by retaining the shape but disregarding the actual size. If the map is really similar to the country it represents, the size of any angle must be accurately reproduced, and, further, the ratio of any two lengths along the ground must be equal to the ratio of the two corresponding lengths on the map, because every length along the ground is reproduced on the map, *reduced in a fixed ratio*, viz. the scale of the map.

The statement that two figures are **congruent** means that the measurement of each element in the first figure (side or angle) is equal to the measurement of the corresponding element in the second figure.

The statement that two figures are **similar** means that each angle of the first figure is equal to the corresponding angle of the second, and, further, that the ratio of the lengths of any two

lines in the first figure is equal to the ratio of the lengths of the two corresponding lines in the second figure.

We obtained the three general tests for congruence of triangles by considering what groups of measurements of the sides and angles must be taken in order to draw elsewhere a copy of a given triangle, identical with it in size and shape.

The object of the next three theorems is to establish what groups of measurements are needed in order to reproduce elsewhere a triangle of the same shape as a given triangle, but of any size.

Suppose ABC is the triangle which is to be reproduced elsewhere in shape, but of any size (*i.e.* on any scale).

(i) Can its shape be reproduced by measuring $\angle$s B, C ?

(ii) Can its shape be reproduced by measuring $\angle$A and the ratio AB : AC ?

(iii) Can its shape be reproduced if you are given the ratio of the three sides AB : BC : CA ?

(iv) Can its shape be reproduced if you are given $\angle$A and the ratio AB : BC ?

The necessary data for reproducing the shape of a triangle give the material for the tests to be applied to two triangles when determining whether they are similar or not. These tests should be investigated orally or experimentally before the formal theorems are established.

EXERCISE LVII.

1. Which of the following pairs of figures must be similar, although unequal in size ;

(i) two circles ; (ii) two equilateral triangles ;
(iii) two isosceles triangles ; (iv) two squares ;
(v) a square and a rectangle ;
(vi) two isosceles right-angled triangles ?

2. A rectangle is 4 in. long, 3 in. wide : another rectangle of the same shape has one side of length 6 inches. What can you say about the length of an adjacent side ?

3. A rectangular sheet of paper is 9″ long, 6″ wide. A line is drawn across it parallel to one of the shorter edges, and 5″ from it,

and the sheet is cut into two pieces along this line. Is either piece the same shape as the original sheet ?

4. A line is drawn through the centre of a rectangle parallel to one of the sides. Will the two portions be the same shape as the original, (i) always, (ii) ever ?

5. Can you draw (i) two triangles, (ii) two quadrilaterals which are equiangular but not similar ?

6. Draw a triangle ABC right-angled at A, but not isosceles ; draw the perpendicular AD from A to BC ; (i) are $\triangle$s ADB, ADC similar ? (ii) are $\triangle$s ABC, ADB similar ?

7. A path 1 yard wide runs all round a rectangular lawn, 20 yd. long, 15 yd. wide. Is the rectangle formed by the outer edge of the path the same shape as the lawn ?

8. Can you draw (i) two triangles, (ii) two quadrilaterals such that the ratios of corresponding sides are all equal but so that the figures are not similar ?

9. ABCD is any quadrilateral ; a line parallel to BC cuts AB at P, DC at Q. Are the quadrilaterals APQD, ABCD (i) equiangular, (ii) similar ?

10. Which of the following pairs of solids must be similar : (i) two spheres ; (ii) two cylinders of the same height ; (iii) two square sheets of paper ; (iv) two regular pyramids (triangular bases) ; (v) a brick and half an equal brick ?

11. A cuboid is $12 \times 8 \times 6$ cm. : another cuboid is of the same shape and has one edge of length 4 cm. What can you say about the lengths of the other two adjacent edges ?

12. Which of the following data will enable you to draw a triangle similar to $\triangle$ABC. If the data are insufficient, state what further information is required.

(i) $\angle A = 72°$, $\angle C = 57°$. (ii) $\angle A = 90°$, AB = 8 cm.

(iii) AB = 6 cm., BC = 9 cm. (iv) $\angle B = 55°$, AB = 2BC.

(v) AB : BC : CA = 2 : 3 : 4. (vi) $\angle C = 63°$, $\frac{AC}{BC} = \frac{5}{7}$.

(vii) $\angle A = 20°$, $\frac{AB}{BC} = \frac{5}{4}$. (viii) $AB = \frac{3}{5}BC$, $AB = \frac{2}{3}AC$.

13. How many distinct facts must be given in order to draw a quadrilateral similar to the quadrilateral ABCD ?

State two different kinds of sets of data which would be sufficient, and then state the corresponding tests for the similarity of two quadrilaterals.

14. The line on a sphere corresponding to a straight line on a plane is a great circle (*e.g.* a meridian circle) : and three such lines form a triangle (*e.g.* two meridians and the equator). Can you draw on a sphere two similar triangles of unequal size ?

THEOREM 55.

If two triangles are equiangular, their corresponding sides are proportional.

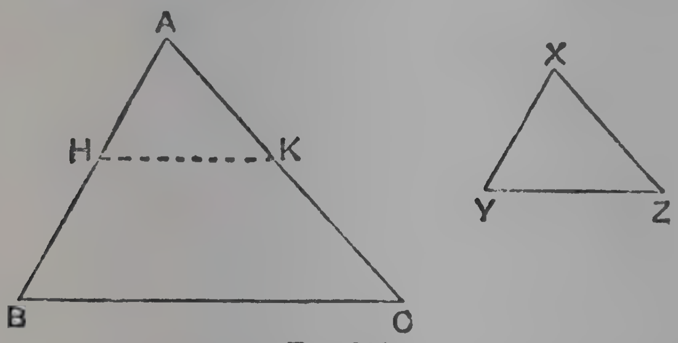

FIG. 296.

Given the triangles ABC, XYZ are equiangular, having $\angle A = \angle X$, $\angle B = \angle Y$, $\angle C = \angle Z$.

To prove $$\frac{AB}{XY} = \frac{AC}{XZ} = \frac{BC}{YZ}.$$

From AB, AC cut off AH, AK equal to XY, XZ. Join HK.

In the $\triangle$s AHK, XYZ, $AH = XY$, constr.

$AK = XZ$, constr.

$\angle HAK = \angle YXZ$, given;

$\therefore$ $\triangle AHK \equiv \triangle XYZ$ (2 sides, inc. angle). $\therefore$ $\angle AHK = \angle XYZ$.

But $\angle XYZ = \angle ABC$, given; $\therefore$ $\angle AHK = \angle ABC$.

But these are corresponding angles, $\therefore$ HK is parallel to BC;

$$\therefore \frac{AB}{AH} = \frac{AC}{AK}.$$

But $AH = XY$ and $AK = XZ$.

$$\therefore \frac{AB}{XY} = \frac{AC}{XZ}.$$

Similarly it can be proved that $\frac{AC}{XZ} = \frac{BC}{YZ}$. Q.E.D.

Definition.

If two polygons are equiangular, and if their corresponding sides are proportional, they are said to be similar.

Theorem 55 proves that equiangular triangles are necessarily similar.

THEOREM 56.

If the three sides of one triangle are proportional to the three sides of the other, then the triangles are equiangular.

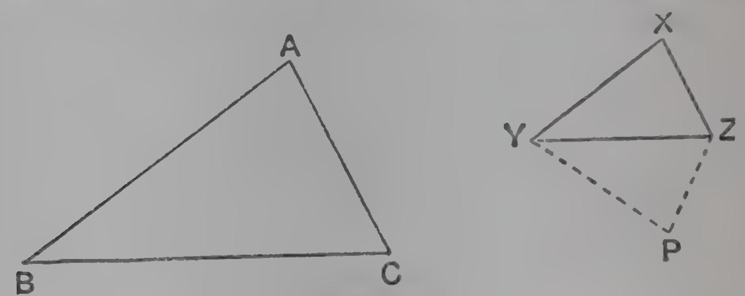

FIG. 297.

Given the $\triangle$s ABC, XYZ are such that $\frac{AB}{XY}=\frac{BC}{YZ}=\frac{CA}{ZX}$.

To prove $\angle A=\angle X$, $\angle B=\angle Y$, $\angle C=\angle Z$.

On the side of YZ opposite to X, draw YP and ZP so that $\angle ZYP=\angle ABC$ and $\angle YZP=\angle ACB$.

Since the $\triangle$s ABC, PYZ are equiangular, by construction

$$\frac{AB}{YP}=\frac{BC}{YZ}.$$

But $$\frac{AB}{XY}=\frac{BC}{YZ}, \text{ given;}$$

$$\therefore \frac{AB}{YP}=\frac{AB}{XY};$$

$$\therefore YP=XY;$$

Similarly $$ZP=XZ;$$

$\therefore$ in the $\triangle$s XYZ, PYZ.

$$XY=PY, \text{ proved.}$$
$$XZ=PZ, \text{ proved.}$$
$$YZ \text{ is common;}$$

$\therefore$ $\triangle XYZ \equiv \triangle PYZ$ (3 sides);

$\therefore$ $\angle XYZ=\angle PYZ$ and $\angle XZY=\angle PZY$.

But $\angle PYZ=\angle ABC$ and $\angle PZY=\angle ACB$, constr.;

$\therefore$ $\angle XYZ=\angle ABC$ and $\angle XZY=\angle ACB$.

$\therefore$ also $\angle YXZ=\angle BAC$. Q.E.D.

THEOREM 57.

If two triangles have an angle of one equal to an angle of the other, and the sides about these equal angles proportional, the triangles are equiangular.

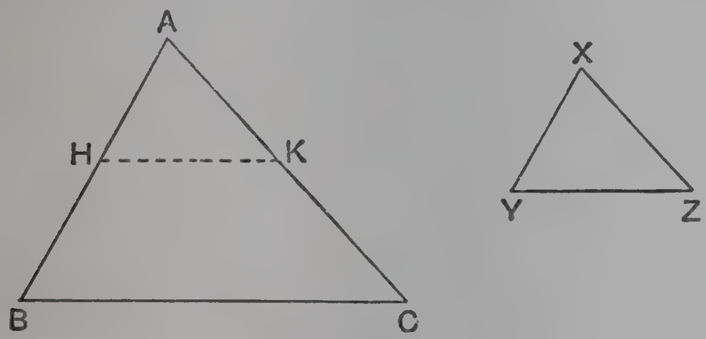

FIG. 298.

Given in the triangles ABC, XYZ, $\angle BAC = \angle YXZ$ and $\frac{AB}{XY} = \frac{AC}{XZ}$

To prove $\angle ABC = \angle XYZ$ and $\angle ACB = \angle XZY$.

From AB, AC, cut off AH, AK equal to XY, XZ. Join HK.

In the $\triangle$s AHK, XYZ,

$$AH = XY, \text{ constr.}$$

$$AK = XZ, \text{ constr.}$$

$$\angle HAK = \angle YXZ, \text{ given;}$$

$$\therefore \triangle AHK \equiv \triangle XYZ \text{ (2 sides, inc. angle);}$$

$$\therefore \angle AHK = \angle XYZ \text{ and } \angle AKH = \angle XZY.$$

Now $\frac{AB}{XY} = \frac{AC}{XZ}$ and $XY = AH$, $XZ = AK$;

$$\therefore \frac{AB}{AH} = \frac{AC}{AK};$$

$$\therefore HK \text{ is parallel to } BC;$$

$$\therefore \angle AHK = \angle ABC \text{ and } \angle AKH = \angle ACB, \text{ corresp. } \angle s$$

But $\angle AHK = \angle XYZ$ and $\angle AKH = \angle XZY$, proved.

$$\therefore \angle ABC = \angle XYZ \text{ and } \angle ACB = \angle XZY.$$

Q.E.D.

EXERCISE LVIII.

1. A pole 10′ high casts a shadow $3\frac{1}{2}'$ long; at the same time a church spire casts a shadow 42′ long. What is its height?

2. In a photograph of a chest of drawers, the height measures 6″ and the breadth 3·2″; if its height is $7\frac{1}{2}$ feet what is its breadth?

3. Show that the triangle whose sides are 5·1″, 6·8″, 8·5″ is right-angled.

4. A halfpenny (diameter 1″) at the distance of 3 yards appears nearly the same size as the sun or moon at its mean distance. Taking the distance of the sun as 93 million miles, find its diameter. Taking the diameter of the moon as 2160 miles, find its mean distance.

5. How far in front of a pinhole camera must a man 6′ high stand in order that a full-length photograph may be taken on a film $2\frac{1}{4}''$ high, $2\frac{1}{2}''$ from the pinhole?

6. The slope of a railway is marked as 1 in 60. What height (in feet) does it climb in $\frac{3}{4}$ mile?

7. A light is 9′ above the floor; a ruler, 8″ long, is held horizontally 4′ above the floor; find the length of its shadow.

8. Two triangles are equiangular; the sides of one are 5″, 8″, 9″; the shortest side of the other is 4 cm.; find its other sides.

9. The bases of two equiangular triangles are 4″, 6″; the height of the first is 5″; find the area of the second.

10. In △ABC, AB=8″, BC=6″, CA=5″; a line XY parallel to BC cuts AB, AC at X, Y; AX=2″; find XY, CY.

11. In quadrilateral ABCD, AB is parallel to DC and AB=8″, AD=3″, DC=5″; AD, BC are produced to meet at P; find PD.

12. A line parallel to BC meets AB, AC at X, Y; BC=8″, XY=5″; the lines BC, XY are 2″ apart. Find the area of △AXY.

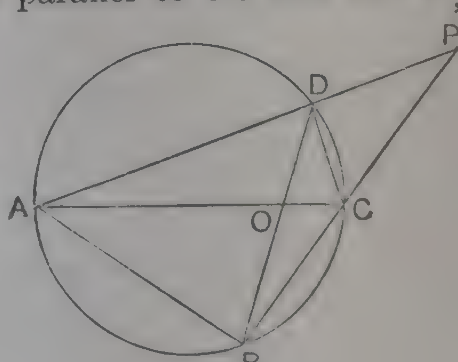

FIG. 299.

13. In Fig. 299,
   (i) if AO=3, OB=2″, AB=4″, DC=$1\frac{1}{2}''$, find CO, DO;
   (ii) if AO=5″, BO=4″, AC=7″, find BD;
   (iii) if PA=9″, PB=8″, AB=4″, PC=3″, find PD, CD;
   (iv) if PA=9″, PB=8″, AC=6″, PC=4″, find BD, PD.

14. Show that the line joining (1, 1) to (4, 2) is parallel to and half of the line joining (0, 0) to (6, 2).

15. Three lines APB, AQC, ARD are cut by two parallel lines PQR, BCD ; AR=3″, RD=2″, BC=4″ ; find PQ.

16. In Fig. 300, AB is parallel to OD ; AB=6′, BO=20′, BE=5′, DQ=9′ ; find OD, BP.

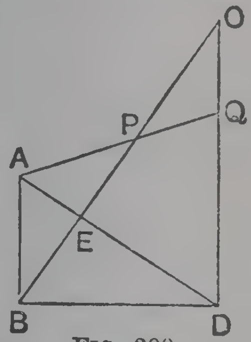

FIG. 300.

17. The diameter of the base of a cone is 9″ and its height is 15″ ; find the diameter of a section parallel to the base and 3″ from it.

18. AXB is a straight line ; AC, XY, BD are the perpendiculars from A, X, B to a line CD ; AC=10, BD=16, AX=12, XB=6 ; find XY.

19. A, B are points on the same side of a line OX and at distances 1″, 5″ from it ; Q and R divide AB internally and externally in the ratio 5 : 3 ; find the distances of Q and R from OX.

20. A rectangular table, 5′ wide, 8′ long, 3′ high, stands on a level floor under a hanging lamp ; the shadow on the floor of the shorter side is 8′ long ; find the length of shadow of the longer side and the height of the lamp above the table.

21. A sphere of 5″ radius is placed inside a conical funnel whose slant side is 12″ and whose greatest diameter is 14″ ; find the distance of the vertex from the centre of the sphere.

22. The length of each arm of a pair of nutcrackers is 6″ ; find the distance between the ends of the arms when a nut 1″ in diameter is placed with its nearer end 1″ from the apex.

23. In Fig. 301, PQBR is a rectangle.

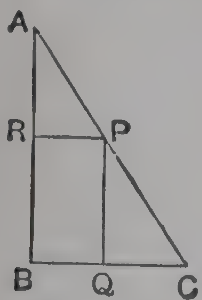

FIG. 301.

(i) If AB=7, PQ=1, PR=2, find BC.

(ii) If AB=7, BC=5, PR=x, PQ=y, find an equation between x, y.

24. In △ABC, ∠ABC=90°, AB=5″, BC=2″ ; the perpendicular bisector of AC cuts AB at Q ; find AQ.

25. The diameter of the base of a cone is 8″; the diameter of a parallel section, 3″ from the base, is 6″; find the height of the cone.

26. In Fig. 302, AB, PN, DC are parallel; AB=4″, BC=5″, CD=3″; calculate PN.

27. ABCD is a quadrilateral such that ∠ABC=90°=∠ACD, AC=5″, BC=3″, CD=10″; calculate the distances of D from BC, BA.

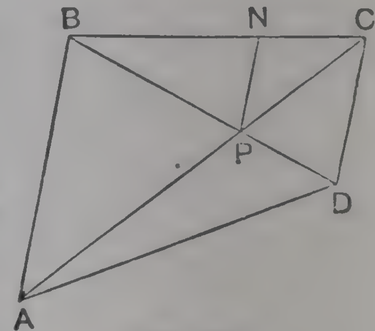

FIG. 302.

28. PQ is a chord of a circle of length 5 cm.; the tangents at P, Q meet at T; PR is a chord parallel to TQ; if PT=8 cm., find PR.

29. (i) A man, standing in a room opposite to and 6′ from a window 27″ wide, sees a wall parallel to the plane of the window. With one eye shut, he can see 18″ less length of wall than with both eyes open; supposing his eyes are 2″ apart, find the distance of the wall from the window and the total length of wall visible.

(ii) If the window is covered by a shutter containing a vertical slit $\frac{1}{2}$″ wide, show that there is a part of the wall out of view which lies between two parts in view and find its length.

(iii) A man in bed at night sees a star pass slowly across a vertical slit in the blind; shortly afterwards, this occurs again. Is it possible that he sees the same star twice? Explain your answer by a figure.

30. A rectangular sheet of paper ABCD is folded so that D falls on B; the crease cuts AB at Q; AB=11″, AD=7″; find AQ.

31. Fig. 303 represents an object HK and its image PQ in a concave mirror, centre O, focus F.

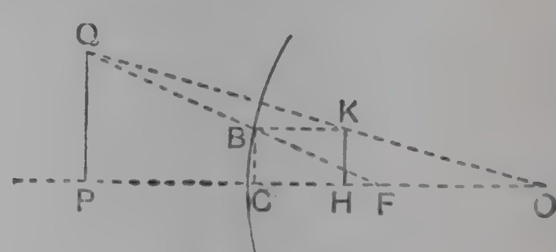

FIG. 303.

$$CH=u,\ CP=v,\ CF=FO=f,\ HK=x,\ PQ=y;$$

prove that (i) $\frac{1}{f}=\frac{1}{u}-\frac{1}{v}$; (ii) $y=\frac{vx}{u}$.

32. In Fig. 304, with the same notation as in Ex. 31, prove that $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$, and find y in terms of x, u, f.

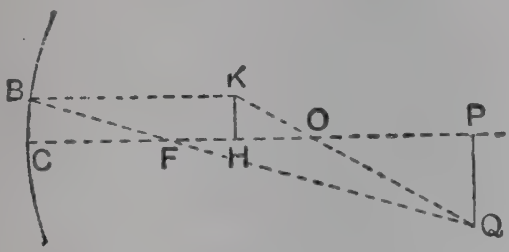

FIG. 304.

33. Fig. 305 represents an object HK and its image PQ in a thin concave lens, centre O, focus F.

$$OH=u, \ OP=v, \ OF=f, \ HK=x, \ PQ=y;$$

prove that (i) $\frac{1}{f}=\frac{1}{v}-\frac{1}{u}$; (ii) $y=\frac{vx}{u}$.

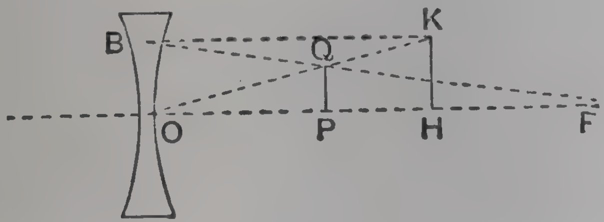

FIG. 305.

34. Fig. 306 represents an object HK and its image PQ in a thin convex lens, centre O, focus F.

$$OH=u, \ OP=v, \ OF=f, \ HK=x, \ PQ=y;$$

prove that $\frac{1}{f}=\frac{1}{u}+\frac{1}{v}$, and find y in terms of x, u, f.

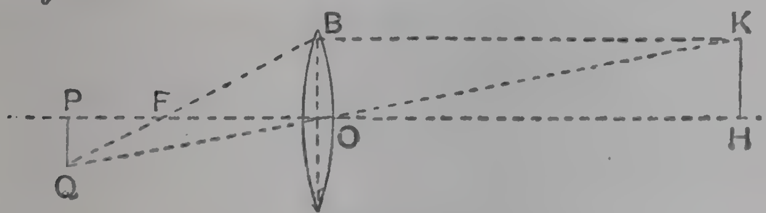

FIG. 306.

35. AB, DC are the parallel sides of a trapezium ABCD; the diagonals cut at O; prove $\frac{AO}{OC}=\frac{AB}{CD}$.

36. BE, CF are altitudes of $\triangle ABC$; prove $\frac{BE}{CF}=\frac{AB}{AC}$.

37. AOB, COD are two intersecting chords of a circle; fill up the blank spaces in (i) $\frac{OA}{AC}=\frac{}{BD}$; (ii) $\frac{OA}{OC}=$—.

38. Two straight lines OAB, OCD cut a circle at A, B, C, D; fill up the blank spaces in (i) $\frac{AC}{BD}=\frac{OA}{}$; (ii) $\frac{OA}{OC}=$—.

39. ABC is a $\triangle$ inscribed in a circle; the bisector of $\angle$BAC cuts BC at Q and the circle at P; prove $\frac{AC}{AP}=\frac{AQ}{AB}$ and complete the equation $\frac{BQ}{AB}=\frac{PC}{}$.

40. In $\triangle$ABC, $\angle BAC=90°$; AD is an altitude; prove that $\frac{DC}{AC}=\frac{AC}{BC}$ and complete the equation $\frac{CD}{DA}=\frac{}{DB}$.

41. The medians BY, CZ of $\triangle$ABC intersect at G; prove that $GY=\frac{1}{3}BY$.

42. BE, CF are altitudes of $\triangle$ABC; prove that $\frac{EF}{BC}=\frac{AF}{AC}$.

43. Two lines AOB, POQ intersect at O; the circles AOP, BOQ cut again at X; prove that $\frac{XA}{XP}=\frac{XB}{XQ}$.

44. Prove that the common tangents of two non-intersecting circles divide (internally and externally) the line joining the centres in the ratio of the radii.

45. M is the mid-point of AB; AXB, MYB are equilateral triangles on opposite sides of AB; XY cuts AB at Z; prove $AZ=2ZB$.

46. AB is a diameter of a circle ABP; PT is the perpendicular from P to the tangent at A; prove $\frac{PT}{PA}=\frac{AP}{AB}$.

47. APB, AQB are two circles; if PAQ is a straight line, prove that $\frac{BP}{BQ}$ equals the ratio of their diameters.

48. ABCD is a parallelogram; any line through C cuts AB produced, AD produced at X, Y; prove $\frac{AD}{BX}=\frac{DY}{AB}$.

49. ABCD is a rectangle; two perpendicular lines are drawn; one cuts AB, CD at E, F; the other cuts AD, BC at G, H; prove $\frac{EF}{GH}=\frac{BC}{AB}$.

50. The diagonals AC, BD of the quadrilateral ABCD meet at O; if the radius of the circle AOD is three times the radius of the circle BOC, prove $AD = 3BC$.

51. ABCD is a parallelogram; P is any point on AB; DP cuts AC at Q; prove $\frac{AP}{AB} = \frac{PQ}{DQ}$.

52. AB, DC are the parallel sides of the trapezium ABCD; any line parallel to AB cuts CA, CB at H, K; DH, DK cut AB at X, Y; prove $AB = XY$.

53. BC, YZ are the bases of two similar triangles ABC, XYZ; AP, XQ are medians; prove $\angle BAP = \angle YXQ$.

54. P is a variable point on a given circle; O is a fixed point outside the circle; Q is a point on OP such that $OQ = \frac{1}{3}OP$; prove that the locus of Q is a circle.

55. In $\triangle ABC$, $\angle BAC = 90°$; ABXY, ACZW are squares outside $\triangle ABC$; BZ, CX cut AC, AB at K, H; prove $AH = AK$.

Congruence and Similarity.

As indicated on p. 237, Theorems 55, 56, 57 give tests for similarity corresponding to the tests for congruence in Theorems 9, 10, 8 respectively.

The special test for congruence in Theorem 12, where two sides and a *not-included angle* are given, may be stated in a more general form, as follows:

In the triangles ABC, PQR,

$$\text{If } \angle ABC = \angle PQR,\ AB = PQ,\ AC = PR,$$

and if *either* (i) both triangles are acute-angled *or* (ii) both triangles are right-angled *or* (iii) both triangles are obtuse-angled, then $\triangle ABC \equiv \triangle PQR$.

The analogous test for similarity is as follows:

In the triangles ABC, PQR,

$$\text{If } \angle ABC = \angle PQR \text{ and } \frac{AB}{AC} = \frac{PQ}{PR} \text{ and if}$$

either (i) both triangles are acute-angled *or* (ii) both triangles are right-angled *or* (iii) both triangles are obtuse-angled, then $\triangle ABC$ is similar to $\triangle PQR$.

THEOREM 58.

If a perpendicular is drawn from the right angle of a right-angled triangle to the hypotenuse, the triangles on each side of the perpendicular are similar to the whole triangle and to one another.

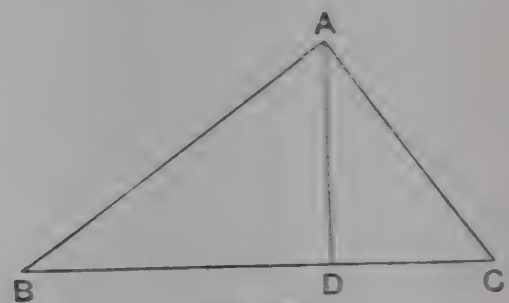

FIG. 307.

Given $\angle BAC = 90°$, and that AD is perpendicular to BC.

To prove $\triangle$s ABC, DBA, DAC are similar.

In $\triangle$s ABC, DBA,

$$\angle BAC = \angle BDA, \text{ right angles.}$$

$$\angle ABC = \angle DBA, \text{ same angle.}$$

$\therefore$ the third angles ACB, DAB are equal, and the triangles are equiangular.

In the same way it may be proved that $\triangle$s ABC, DAC are equiangular.

$\therefore$ the three triangles DBA, ABC, DAC are equiangular and therefore similar. Q.E.D.

Corollary 1. *The square on the perpendicular is equal to the rectangle contained by the segments of the base*

$$AD^2 = BD \,.\, DC.$$

For, since $\triangle$s DBA, DAC are similar,

$$\frac{DA}{DB} = \frac{DC}{DA}, \quad \therefore \; DA^2 = DB \,.\, DC.$$

Corollary 2. *The square on either of the sides containing the right angle is equal to the rectangle contained by the hypotenuse and the segment of the hypotenuse adjacent to that side*

$$BA^2 = BC . BD \text{ and } CA^2 = CB . CD.$$

For, since $\triangle$s ABC, DBA are similar,

$$\frac{BA}{BD} = \frac{BC}{BA}, \quad \therefore \; BA^2 = BC . BD,$$

and, similarly, $CA^2 = CB . CD$.

Alternative methods of proof of these corollaries have been given on page 217. The deduction of Corollary 2 from the ordinary proof of Pythagoras' theorem is especially important. But, alternatively, the above method of proof may also be used to establish Pythagoras' theorem (this is the method adopted in French text-books).

$$BA^2 = BC . BD \text{ and } CA^2 = CB . CD.$$

$$\therefore \; BA^2 + CA^2 = BC . BD + CB . CD = BC(BD + DC) = BC . BC = BC^2.$$

Mean Proportional.

If a, b, c are three quantities such that $a : b = b : c$ *or* $\frac{a}{b} = \frac{b}{c}$ *or* $ac = b^2$, then b is called the **mean proportional** between a and c.

In Fig. 307, AD is the mean proportional between BD and DC, because $AD^2 = BD . DC$, and BA is the mean proportional between BD and BC, because $BA^2 = BD . BC$.

CONSTRUCTION 23.

Construct a mean proportional to two given lines.

Given two lines of lengths a, b units.

To construct a line of length x units such that $\frac{a}{x}=\frac{x}{b}$ or $x^2=ab$.

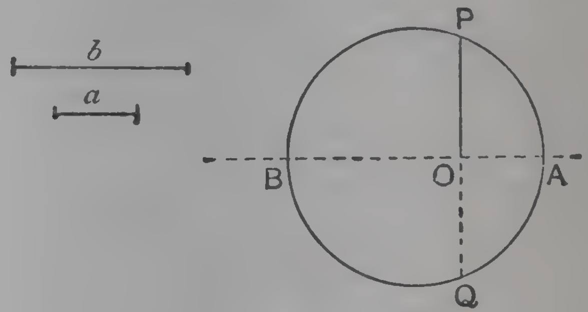

FIG. 303.

METHOD I. Take a point O on a line and cut off from the line on *opposite* sides of O, parts OA, OB of lengths a, b units.

On AB as diameter, describe a circle.

Draw OP perpendicular to AB to cut the circle at P.

Then OP is the required mean proportional.

Proof. Produce PO to meet the circle at Q.

PQ is a chord perpendicular to the diameter AB.

$$\therefore \text{PO}=\text{OQ}.$$

But PO . OQ = AO . OB, intersecting chords of a circle.

$$\therefore \text{OP}^2=a\,.\,b,$$

$$\text{or } \frac{a}{\text{OP}}=\frac{\text{OP}}{b}.$$

Q.E.F.

Note.—This should be compared with Constr. 20, p. 221.

METHOD II. Take a point O on a line and cut off from the line on the *same* side of O, parts OA, OB of lengths a, b units.

On OB as diameter, describe a circle.

Draw AQ perpendicular to OB to meet the circle at Q.

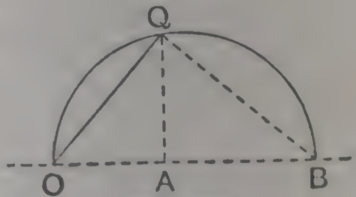

FIG. 309.

Join OQ. Then OQ is the required mean proportional.

Proof. $\angle OQB = 90°$; angle in semicircle.

$\therefore$ OQ is a tangent to the circle on QB as diameter.

But $\angle QAB = 90°$, $\therefore$ circle on QB as diameter passes through A.

$$\therefore\ OQ^2 = OA\,.\,OB, \text{ tangent property of circle.}$$

$$\therefore\ OQ^2 = a\,.\,b \ \text{ or } \ \frac{a}{OQ} = \frac{OQ}{b}.$$

Q.E.F.

Note.—In practical constructions Method II. is often preferable to Method I.

These constructions may also be proved by quoting the Corollaries of Theorem 58.

EXERCISE LIX.

1. Construct a mean proportional between 5 and 8; measure it.

2. Construct a line of length $\sqrt{43}$ cms. (Don't take a mean between 1 and 43, this leads to inaccurate drawing; take numbers closer together, such as 5 and 8·6, $\frac{43}{5} = 8{\cdot}6$.)

3. Find graphically $\sqrt{37}$.

4. Solve graphically the equation $(x-3)^2 = 19$.

5. Draw a rectangle of sides 4 cm., 7 cm., and construct a square of equal area; measure its side.

6. Construct a square equal in area to an equilateral triangle of side 5 cm.; measure its side.

7. Construct a square equal in area to a quadrilateral ABCD given AB = BC = 4, CD = 6, DA = 7, AC = 6 cm.; measure its side.

8. Draw a line AB; construct a point P on AB such that $AP^2 = \frac{2}{3}AB^2$.

9. Draw a circle, centre O; construct a concentric circle whose area is one-third of the first circle.

THEOREM 50. (Second Proof.)

If two chords of a circle (produced if necessary) cut one another, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

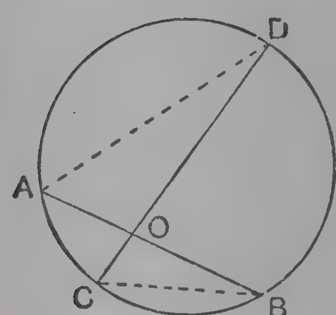

FIG. 310 (1).

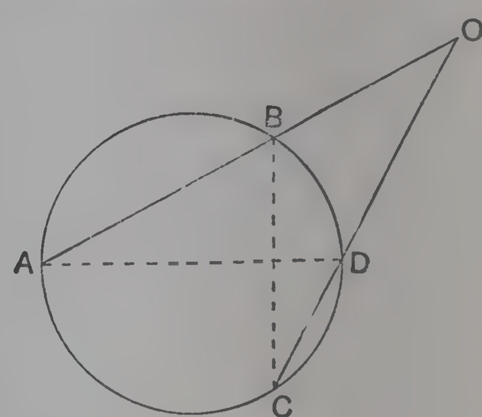

FIG. 310 (2).

(1) *Given* two chords AB, CD intersecting at O.

To prove OA . OB = OC . OD.

Join BC, AD.

In the $\triangle$s AOD, BOC,

$\angle$OAD = $\angle$OCB, in the same segment, Fig. 310 (1) and Fig. 310 (2).

$\angle$AOD = $\angle$COB, vert. opp. in Fig. 310 (1),
same $\angle$ in Fig. 310 (2).

$\therefore$ the third $\angle$ODA = the third $\angle$OBC.

$\therefore$ triangles are equiangular.

$$\therefore \frac{OA}{OC} = \frac{OD}{OB}.$$

$\therefore$ OA . OB = OC . OD. Q.E.D.

THEOREM 51. (Second Proof.)

If from any point outside a circle, a secant and a tangent are drawn, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.

Given a chord AB meeting the tangent at T in O.

To prove $OA \cdot OB = OT^2$.

Join AT, BT.

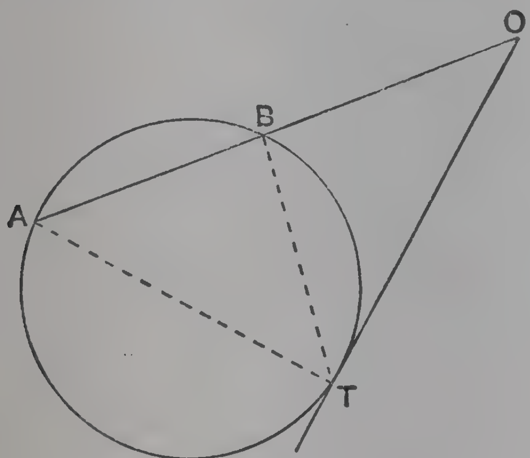

FIG. 311.

In the $\triangle$s AOT, TOB,

$\angle TAO = \angle BTO$, alt. segment.

$\angle AOT = \angle TOB$, same angle.

$\therefore$ the third $\angle ATO =$ the third $\angle TBO$.

$\therefore$ the triangles are equiangular.

$$\therefore \frac{OA}{OT} = \frac{OT}{OB}. \qquad \therefore OA \cdot OB = OT^2.$$

Q.E.D.

Note.—This may also be deduced from Theorem 50 by taking the limiting case when D coincides with C in Fig. 310 (2).

The converse properties are as follows:

(i) If two lines AOB, COD are such that $AO \cdot OB = CO \cdot OD$, then A, B, C, D lie on a circle.

(ii) If two lines OBA, ODC are such that $OA \cdot OB = OC \cdot OD$, then A, B, C, D lie on a circle.

(iii) If two lines OBA, OT are such that $OA \cdot OB = OT^2$, then the circle through A, B, T touches OT at T.

These are proved easily by a *reductio ad absurdum* method.

EXERCISE LX.

Note.—For additional examples and riders on the rectangle properties of a circle, see p. 218.

1. Find a mean proportional between (i) 3 and 48; (ii) $12x$, $3xy^2$.

2. From a point P on a circle, PN is drawn perpendicular to a diameter AB; AN=3″, NB=12″; find PN.

3. In △ABC, ∠BAC=90°; AD is an altitude; AB=5″, AC=12″; find BD.

4. In △ABC, AB=8, AC=12; a circle through B, C cuts AB, AC at P, Q; BP=5; find CQ.

5. The diagonals of a cyclic quadrilateral ABCD meet at O; AC=9, BD=12, OA=4; find OB.

6. In Fig. 312,
  (i) If AB=9, BO=3, find OT.
  (ii) If OB=6, OT=12, find AB.
  (iii) If OA=3, AB=2, AT=4, find BT.
  (iv) If AB=8, AT=6, BT=5, find OT.

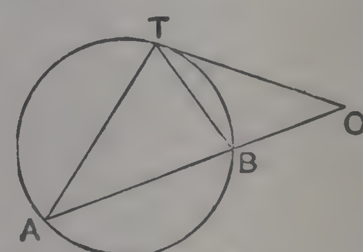

FIG. 312.

7. ABC is a triangle inscribed in a circle; AB=AC=10″, BC=12″; AD is drawn perpendicular to BC and is produced to meet the circle in E; find DE and the radius of the circle.

8. In △ABC, ∠ABC=90°, AB=3″, BC=4″; find the radius of the circle which passes through A and touches BC at C.

9. In △ABC, ∠BAC=90°; AD is an altitude; BC=a, CA=b, AB=c, AD=h, BD=x, DC=y; prove that (i) $h^2=xy$; (ii) $b^2=y(x+y)$; (iii) $hc=bx$; (iv) $\dfrac{b^2}{c^2}=\dfrac{y}{x}$.

10. In Fig. 312, if OA=2OT, prove AB=3BO.

11. AOB, COD are two perpendicular chords of a circle, centre K; AO=6, CO=10, OD=12; find OK, AK.

12. X is the mid-point of a line TY of length 2″; TZ is drawn so that ∠ZTX=45°; a circle is drawn through X, Y touching TZ at P; prove ∠TXP=90°, and find the radius of the circle.

13. ABC is a △ inscribed in a circle; the tangent at C meets AB produced in D; BC=p, CA=q, AB=r, BD=x, CD=y; find x, y in terms of p, q, r.

14. Express, in the form of equal ratios, the equations: (i) $xy=ab$; (ii) $pq=r^2$; (iii) $OA . OB=OC . OD$; (iv) $ON . OT=OP^2$.

15. The diagonals of a cyclic quadrilateral ABCD intersect at O; prove $AD . OC=BC . OD$.

16. Two lines OAB, OCD cut a circle at A, B, C, D; prove $OA . BC=OC . AD$.

17. Two chords AB, CD of a circle intersect at O; if D is the mid-point of arc AB, prove $CA . CB=CO . CD$.

18. In $\triangle ABC$, $AB=AC$; D is a point on AC such that $BD=BC$; prove $BC^2=AC . CD$.

19. ABCD is a cyclic quadrilateral; P is a point on BD such that $\angle PAD=\angle BAC$; prove that (i) $BC . AD=AC . DP$; (ii) $AB . CD=AC . BP$; (iii) $BC . AD+AB . CD=AC . BD$.

20. AB is a diameter of a circle, centre O; AP, PQ are equal chords; prove $AP \cdot PB=AQ . OP$.

21. AD is an altitude of $\triangle ABC$; prove that the radius of the circle ACB equals $\frac{AB . AC}{2AD}$. (Draw diameter through A.)

22. A line PQ is divided at R so that $PR^2=PQ . RQ$; TQR is a $\triangle$ such that $TQ=TR=PR$; prove $PT=PQ$.

23. PQR is a $\triangle$ inscribed in a circle; the tangent at P meets QR produced at T; prove $\frac{TQ}{TR}=\frac{PQ^2}{PR^2}$.

24. In $\triangle ABC$, $\angle BAC=90°$; E is a point on BC such that $AE=AB$; prove $BE . BC=2AE^2$.

25. AD is an altitude of $\triangle ABC$; if $AB . BC=AC^2$ and if $AB=CD$, prove $\angle BAC=90°$.

26. Two chords AB, AC of a circle are produced to P, Q so that $AB=BP$ and $AC=CQ$; if PQ cuts the circle at R, prove $AR^2=PR . RQ$.

27. The tangent at a point C on a circle is parallel to a chord DE and cuts two other chords PD, PE at A, B; prove $\frac{AC}{CB}=\frac{AD}{BE}$.

28. AB is a diameter of a circle, centre O; the tangents at A, B meet any other tangent at H, K; prove $AH . BK=AO^2$.

29. ABC is a $\triangle$ inscribed in a circle; a line through B parallel to AC cuts the tangent at A in P; a line through C parallel to AB cuts AP in Q; prove $\frac{AP}{AQ}=\frac{AB^2}{AC^2}$.

30. AB is a chord of a circle APB; the tangents at A, B meet at T; PH, PK, PZ are the perpendiculars to TA, TB, AB; prove

$$PH . PK=PZ^2.$$

THEOREM 59.

The ratio of the areas of two similar triangles is equal to the ratio of the squares on corresponding sides.

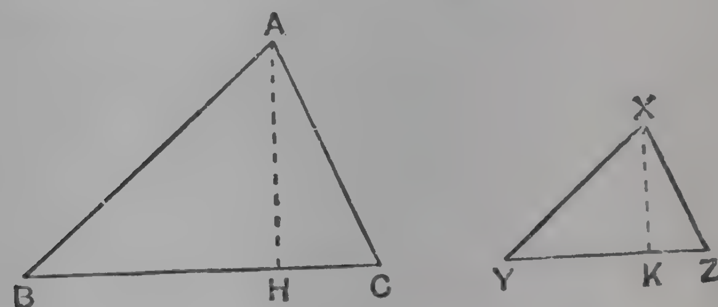

FIG. 313.

Given the triangles ABC, XYZ are similar.

To prove $\dfrac{\triangle ABC}{\triangle XYZ}=\dfrac{BC^2}{YZ^2}$.

Draw the altitudes AH, XK.

In the $\triangle$s AHB, XKY,

$\angle ABH = \angle XYK$, given.

$\angle AHB = \angle XKY$, rt. $\angle$s constr.

$\therefore$ the third $\angle BAH =$ the third $\angle YXK$.

$\therefore$ the $\triangle$s AHB, XKY are similar.

$$\therefore \frac{AH}{XK}=\frac{AB}{XY}.$$

But $\dfrac{AB}{XY}=\dfrac{BC}{YZ}$, since $\triangle$s ABC, XYZ are similar.

$$\therefore \frac{AH}{XK}=\frac{BC}{YZ}.$$

But $\triangle ABC = \frac{1}{2}AH \,.\, BC$ and $\triangle XYZ = \frac{1}{2}XK \,.\, YZ$.

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{AH \,.\, BC}{XK \,.\, YZ}.$$

But $\dfrac{AH}{XK}=\dfrac{BC}{YZ}$, proved.

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{BC^2}{YZ^2}.$$

Q.E.D.

If two polygons are similar, it can be proved that they can be divided up into the same number of similar triangles.

Hence it follows that the ratio of the areas of two similar polygons is equal to the ratio of the squares on corresponding sides.

The following facts are also of importance (see Ex. 28, 29) :

(i) The ratio of the areas of the surfaces of similar solids equals the ratio of the squares of their linear dimensions.

(ii) The ratio of the volumes of similar solids equals the ratio of the cubes of their linear dimensions.

EXERCISE LXI.

1. A screen, 6′ high (not necessarily rectangular), requires 27 sq. ft. of material for covering ; how much is needed for a screen of the same shape, 4′ high ?

2. On a map whose scale is 6″ to the mile, a plot of ground is represented by a triangle of area $2\frac{1}{4}$ sq. inches ; what is the area (in acres) of the plot ?

3. The sides of a triangle are 6 cm., 9 cm., 12 cm. ; how many triangles whose sides are 2 cm., 3 cm., 4 cm. can be cut out of it ? How would you cut it up ?

4. Show how to divide any triangle into 25 triangles similar to it.

5. The area of the top of a table, 3 feet high, is 20 sq. ft.; the area of its shadow on the floor is 45 sq. ft. ; find the height of the lamp above the floor.

6. A light is 12 feet above the ground ; find the area of the shadow of the top of a table 4 ft. high, 9 ft. long, 5 ft. broad.

7. ABC, XYZ are similar triangles ; AD, XK are altitudes ; AB=15, BC=14, CA=13, AD=12, XY=5 ; find XK and the ratio of the areas of $\triangle$s ABC, XYZ.

8. A triangle ABC is divided by a line HK parallel to BC into two parts AHK, HKCB of areas 9 sq. cm., 16 sq. cm. ; BC=7 cm. ; find HK.

9. ABC is a $\triangle$ such that AB=AC=2BC ; D is a point on AC such that $\angle$DBC=$\angle$BAC ; a line through D parallel to BC cuts AB in E ; find the ratio of the areas $\triangle$ABC : $\triangle$BCD : $\triangle$BED : $\triangle$EDA.

10. Water in a supply pipe of diameter 1 ft. comes out through a tap 3″ in diameter : in the pipe it is moving at 5″ a second ; with what velocity does it come out of the tap ?

11. If it costs £3 to gild a sphere of radius 3 ft., what will it cost to gild a sphere of radius 4 ft. ?

12. Two hot-water cans are the same shape: the smaller is 9″ high and holds a quart; the larger is 15″ high; how much will it hold ?

13. How many times can a cylindrical tumbler 4″ high and 3″ in diameter be filled from a cylindrical cask 40″ high and 30″ in diameter ?

14. A metal sphere, radius 3″, weighs 8 lb.; find the weight of a sphere of the same metal 1′ in radius.

15. A cylindrical tin 5″ high holds $\frac{1}{4}$ lb. of tobacco; how much will a tin of the same shape 8″ high hold ?

16. Two models of the same statue are made of the same material; one is 3″ high and weighs 8 oz.; the other weighs 4 lb.; what is its height ?

17. A lodger pays 8 pence for a scuttle of coal, the scuttle being 20″ deep; what would he pay if the scuttle was the same shape and $2\frac{1}{2}$ feet deep.

18. A tap can fill half of a spherical vessel, radius $1\frac{1}{2}$ feet, in 2 minutes; how long will two similar taps take to fill one-quarter of a spherical vessel of radius 4 feet ?

19. Two leaden cylinders of equal lengths and diameters 3″, 4″ are melted and recast as a single cylinder of the same length; what is its diameter ?

20. In Fig. 314, not drawn to scale, the lines AB, CD bisect each other at right angles; AB=6 cm., CD=4 cm., PAQ, RBS are arcs of circles of radii 1 cm.; PCS, QDR are arcs of circles of radii $3\frac{1}{2}$ cm. touching the former arcs. Construct a similar figure in which the length of the line corresponding to AB is 9 cm.

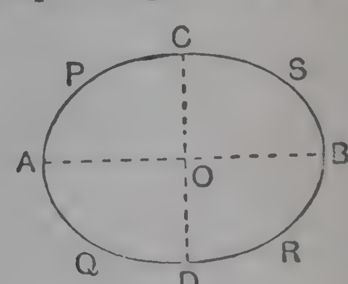

FIG. 314.

The area of the first figure is approximately 18 sq. cms.; what is the area of the enlarged figure ?

If in the given figure, the curve is rotated about AB to form an egg-shaped solid, its volume is approximately 48 c.c.; what is the volume of the solid obtained similarly from the enlarged figure ?

21. The sides of a $\triangle$ABC are trisected as in the figure; prove that the area of PQRSXY$=\frac{2}{3}\triangle$ABC.

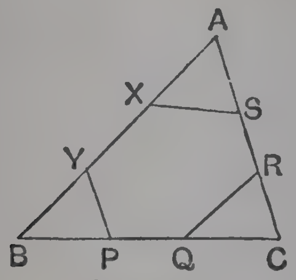

FIG. 315.

22. If in the $\triangle$s ABC, XYZ, $\angle$BAC$=\angle$YXZ, prove that

$$\frac{\triangle ABC}{\triangle XYZ}=\frac{AB \cdot AC}{XY \cdot XZ}.$$

23. Two lines OAB, OCD meet a circle at A, B, C, D; prove that $\frac{\triangle OAD}{\triangle OBC}=\frac{AD^2}{BC^2}$. What result is obtained by making B coincide with A?

24. In $\triangle$ABC, $\angle$BAC$=90°$ and AD is an altitude; prove

$$\frac{AB^2}{AC^2}=\frac{BD}{DC}.$$

25. ABCD is a parallelogram; P, Q are the mid-points of CB, CD; prove $\triangle$APQ$=\frac{3}{8}$ parallelogram ABCD.

26. In $\triangle$ABC, $\angle$BAC$=90°$ and AD is an altitude; DE is the perpendicular from D to AB; prove $\frac{BE}{BA}=\frac{BA^2}{BC^2}$.

27. In $\triangle$ABC, $\angle$BAC$=90°$; BCX, CAY, ABZ are similar triangles with X, Y, Z corresponding points; prove $\triangle$CAY$+\triangle$ABZ$=\triangle$BCX.

28. If x in. is the length of some definite dimension in a figure of given shape, its area$=kx^2$ sq. in. where k is constant for different sizes. Find k for (i) square, side x; (ii) square, diagonal x; (iii) circle, radius x; (iv) circle, perimeter x; (v) equilateral triangle, side x; (vi) regular hexagon, side x; (vii) surface of cube, side x; (viii) surface of sphere, radius x.

29. If x in. is the length of some definite dimension in a figure of given shape, its volume$=kx^3$ cu. in. where k is constant for different sizes. Find k for (i) cube, edge x; (ii) cube, diagonal x; (iii) sphere, diameter x; (iv) sphere, equator x; (v) the greatest circular cylinder that can be cut from a cube, edge x; (vi) circular cone, vertical angle 90°, height x; (vii) regular tetrahedron, edge x.

ILLUSTRATIVE CONSTRUCTIONS.

I. To construct a pentagon similar to a given pentagon and such that corresponding sides are in a given ratio.

Given a pentagon OABCD and a ratio XY : XZ.

To construct a pentagon OA′B′C′D′ such that

$$\frac{OA'}{OA}=\frac{A'B'}{AB}=\ldots=\frac{XY}{XZ}.$$

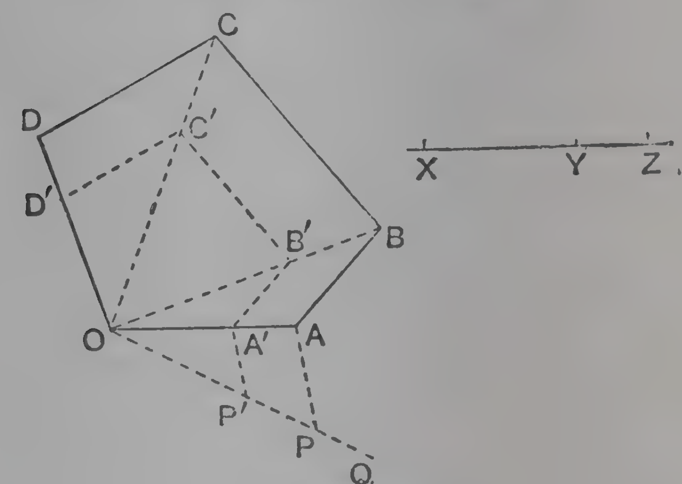

FIG. 316.

Join OB, OC.

Draw any line OQ and cut off parts OP′, OP equal to XY, XZ. Join PA.

Through P′ draw P′A′ parallel to PA to meet OA at A′.

Through A′ draw A′B′ parallel to AB to meet OB at B′.

Through B′ draw B′C′ parallel to BC to meet OC at C′.

Through C′ draw C′D′ parallel to CD to meet OD at D′.

Then OA′B′C′D′ is the required pentagon.

Proof. Since A′B′ is parallel to AB, $\triangle$s OA′B′, OAB are similar,

$$\therefore \frac{OA'}{OA}=\frac{A'B'}{AB}=\frac{OB'}{OB}.$$

Similarly $$\frac{OB'}{OB}=\frac{B'C'}{BC}=\frac{OC'}{OC}, \text{ and so on.}$$

$$\therefore \frac{OA'}{OA}=\frac{A'B'}{AB}=\frac{B'C'}{BC}=\frac{C'D'}{CD}=\frac{D'O'}{DO}.$$

Also $$\frac{OA'}{OA}=\frac{OP'}{OP}=\frac{XY}{XZ}.$$

$\therefore$ the sides of $OA'B'C'D'$ are proportional to the sides of $OABCD$ in the ratio $XY : XZ$.

Further, by parallels, the pentagons are equiangular.

$\therefore$ the pentagons are similar and their corresponding sides are in the given ratio. Q.E.F.

EXERCISE LXII.

1. Given a triangle ABC, construct a point P on BC such that the lengths of the perpendiculars from P to AB and AC are in the ratio 2 : 3.

2. ABC is an equilateral triangle of side 5 cm., construct a point P inside it such that the perpendiculars from P to BC, CA, AB are in the ratio 1 : 2 : 3. Measure AP.

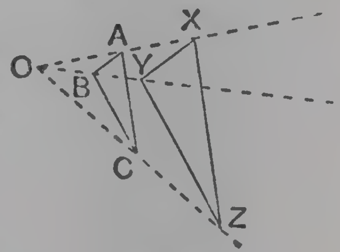

FIG. 317.

3. Draw any triangle ABC, use the method indicated in Fig. 317 to construct a triangle XYZ similar to triangle ABC and such that XY=2AB.

4. Given a quadrilateral ABCD, construct a similar quadrilateral each side of which is $\frac{5}{3}$ of the corresponding side of ABCD.

5. Given a triangle ABC and its median AD, construct a similar triangle XYZ and its median XW, such that $XW=\frac{2}{3}AD$.

6. Construct an equilateral triangle such that the length of the line joining one vertex to a point of trisection of the opposite side is 2″; measure its side.

7. Construct a square ABCD, given that the length of the line joining A to the mid-point of BC is 3″; measure its side.

8. Construct a triangle ABC, given $\angle BAC=54°$, $\angle ABC=48°$, and the sum of the three medians is 12 cm. Measure AB.

9. Inscribe in a given triangle a triangle whose sides are parallel to the sides of another given triangle.

10. Given two radii OA, OB of a circle, centre O; construct a square such that one vertex lies on OA, one vertex on OB, and the remaining vertices on the arc AB.

11. Inscribe a regular octagon in a square.

12. Construct a circle to touch two given lines and a given circle, centre O, radius a. (Draw two lines parallel to the given lines at a distance a from them: construct a circle to touch these lines and pass through O. Its centre is the centre of the required circle.)

13. Draw a line AB and take a point O 1″ from it; P is a variable point on AB; Q is a point such that OQ=OP and ∠POQ=50°. Construct the locus of Q. (The locus of Q is obtained by revolving AB about O through 50°.)

14. ABC is a given triangle; P is a variable point on BC; Q is a point such that the triangles ABC, APQ are similar. Construct the locus of Q. (Use the idea of Ex. 13.)

15. APQ is a triangle of given shape; A is a fixed point, P moves on a fixed circle; construct the locus of Q. (Use the idea of Ex. 13.)

II. Construct a circle to pass through two given points and touch a given line.

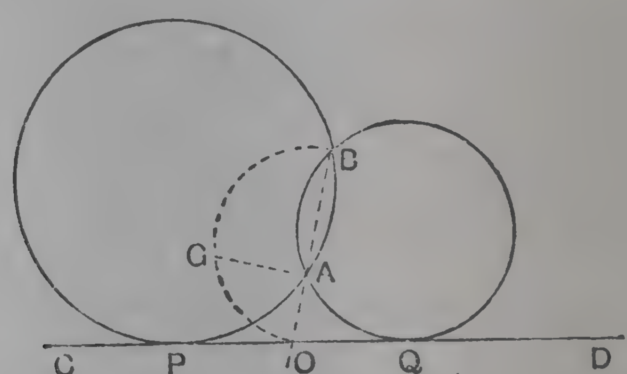

FIG. 318.

Given two points A, B, and a line CD.

To construct a circle to pass through A, B and touch CD.

Join AB and produce it to meet CD at O.

Construct the mean proportional OG to OA, OB, and cut off from CD on each side of O parts OP, OQ equal to OG.

Construct the circles through A, B, P and A, B, Q.

These are the required circles.

Proof. Since $OA \cdot OB = OG^2 = OP^2 = OQ^2$,

OP, OQ are tangents to the circles ABP, ABQ.

Q.E.F.

Note that the method fails if AB is parallel to CD. This special case forms an easy exercise.

III. Bisect a triangle by a line parallel to one side.

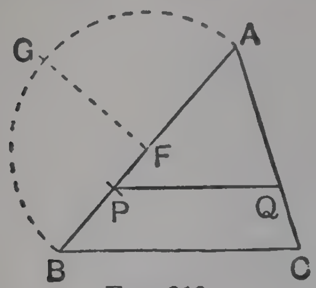

FIG. 319.

Given a triangle ABC.

To construct a line parallel to BC, cutting AB, AC at P, Q so that PQ bisects $\triangle$ABC.

Bisect AB at F.

Construct the mean proportional AG between AF, AB.

From AB cut off AP equal to AG.

Draw PQ parallel to BC, cutting AC at Q.

Then PQ is the required line.

Proof. $\dfrac{\triangle APQ}{\triangle ABC} = \dfrac{AP^2}{AB^2} = \dfrac{AF \cdot AB}{AB^2} = \dfrac{AF}{AB} = \frac{1}{2}.$

EXERCISE LXIII.

1. Given a quadrilateral ABCD, construct a similar quadrilateral with its area $\frac{2}{5}$ of the area of ABCD.

2. Given a triangle ABC, construct an equilateral triangle of equal area.

3. Given three lines whose lengths are a, b, c cm., construct a line of length x cm. such that $\dfrac{x}{a} = \dfrac{b^2}{c^2}$.

4. Given two equilateral triangles, construct an equilateral triangle whose area is the sum of their areas.

5. Construct a circle to pass through two given points A, B and touch a given line CD.

Use the method indicated in Fig. 320 and obtain two solutions.

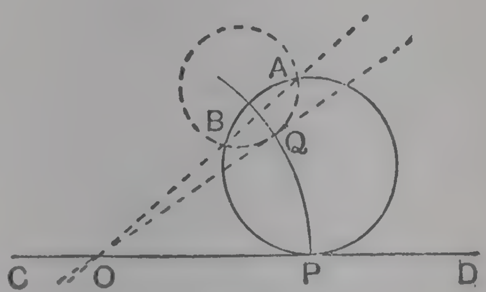

FIG. 320.

6. Construct a circle to pass through two given points A, B and to touch a given circle.

7. Solve graphically the equations $x-y=5$, $xy=16$.

8. OA, OB are two lines such that OA=6 cm., $\angle$AOB=40°; construct a circle touching OA at A and intercepting on OB a length of 5 cm.

9. Construct a circle to pass through a given point, touch a given circle and have its centre on a given line.

10. Given three circles, each external to the others, construct a point such that the tangents from it to the three circles are of equal length.

11. Draw a circle of radius 5 cm. and take a point A 3 cm. from the centre; construct a chord PQ of the circle passing through A such that PA=$\frac{2}{3}$AQ.

MISCELLANEOUS CONSTRUCTIONS. IV.

EXERCISE LXIV.

1. Draw a line AB; if AB is of length x inches, construct a line of length x^2 inches.

2. ABC is a given equilateral triangle of side 5 cm.; construct a line outside it such that the perpendiculars from A, B, C to the line are in the ratio 2 : 3 : 4, and measure the last.

3. Construct a triangle ABC, given $\angle$BAC=48°, $\angle$BCA=73°, and the median BE=5 cm.; measure AC.

4. Construct a triangle ABC, given $\angle$ABC=62°, $\angle$ACB=75°, and AB−BC=2 cm.; measure BC.

5. Inscribe in a given triangle a rectangle having one side double the other.

6. Draw a triangle of sides 5, 6, 7 cms. and construct a square of equal area; measure its side. Check your result from the formula $\sqrt{s(s-a)(s-b)(s-c)}$.

7. Divide a square into three parts of equal area by lines parallel to one diagonal.

8. Given two lines AB and CD, construct a point P on AB produced such that PA . PB=CD².

REVISION PAPERS.

SECTIONS I.-VI.

57.

1. (i) How many angles each greater than 170° is it possible for a ten-sided convex polygon to have ?

(ii) What is the least number of obtuse angles a ten-sided convex polygon must have ?

2. Fig. 321 is a plan of a tennis court, the given measurements are in feet. The only corner-markers that are visible are those at P and Q. Give the least necessary calculations you must make to find the exact position of A with the aid only of two tape measures.

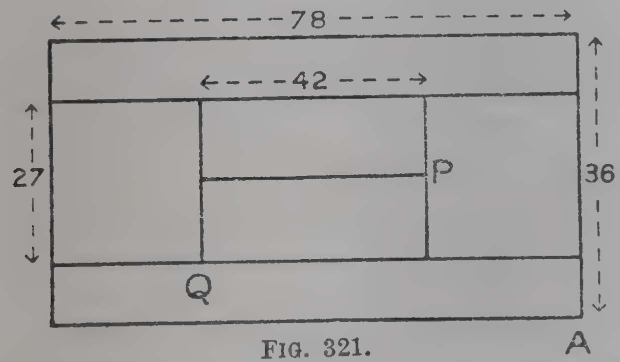

Fig. 321.

3. If, in Fig. 322, PT is a tangent, prove that the circle through P, T, R must pass through B.

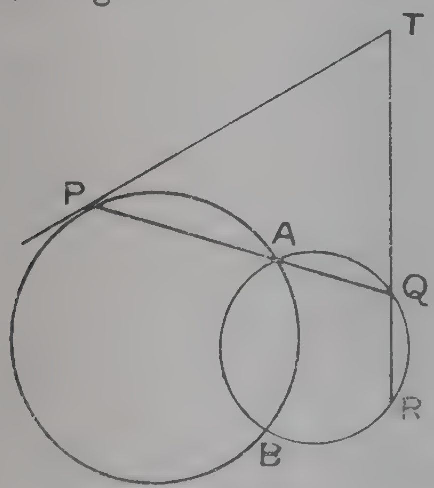

Fig. 322.

4. XAY is a diameter of a circle, centre A ; Z is the middle point of AY. If a circle is drawn on XZ as diameter, prove that the length

of the tangent to this circle from any point P on the outer circle is equal to $\frac{1}{2}$PX. (Let PX cut the inner circle at Q, join QZ, PY.)

58.

1. Prove that the marked angles in Fig. 323 are such that $a+b=x+y+z$.

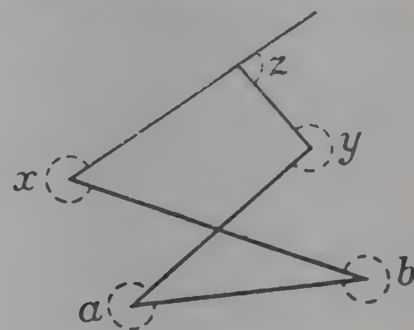

FIG. 323.

2. K is any point on the diameter AB of a circle; P is the mid-point of the arc AB; prove that $AK^2+KB^2=2PK^2$.

3. Fig. 324 represents parts of two circles which touch at a point A on CB produced; the lines CB and QP when produced intersect at right angles at the centre of the larger circle. Given PQ=3″, BC=5″, calculate the radius of each circle.

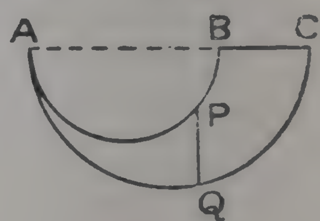

FIG. 324.

4. ABC is an equilateral triangle; BC is produced each way to P, Q; if $\angle PAQ=120°$, prove that the triangles PBA, ACQ are similar; hence show that $PB \cdot CQ=BC^2$ and $\frac{PB}{CQ}=\frac{AP^2}{AQ^2}$.

59.

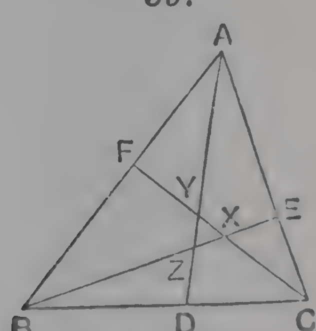

FIG. 325.

1. In Fig. 325, if $\angle ADC=\angle BEA=\angle CFB$, prove that the triangles ABC, XYZ are equiangular.

2. The tangent at a point R of a circle meets a chord PQ at T; O is the centre; E is the mid-point of PQ; prove $\angle ROT = \angle RET$.

3. A line AB, 8 cm. long, is divided internally and externally in the ratio 3 : 1 at P, Q respectively; find PQ : AB.

4. ABCD is a quadrilateral; a line AF parallel to BC meets BD at F; a line BE parallel to AD meets AC at E; prove EF is parallel to CD.

60.

1. The sides AB, BC, CA of $\triangle ABC$ are produced their own lengths to X, Y, Z; prove $\triangle XYZ = 7\triangle ABC$.

2. ABCD is a quadrilateral; the circles on AB, BC as diameters intersect again at P; the circles on AD, DC as diameters intersect again at Q; prove BP is parallel to DQ.

3. A town occupies an oval area of length 2400 yards, breadth 1000 yards: a plan is made of it on a rectangular sheet of paper 18″ long, 12″ wide. What is the best scale to choose?

4. ABC is a triangle inscribed in a circle; AD is an altitude; AP is a diameter; prove $\frac{AB}{AP} = \frac{AD}{AC}$ and complete the equation $\frac{BD}{AB} = \frac{\quad}{AP}$.

61.

1. AB is a diameter of a circle; AOC, BOE are two chords such that $\angle CAB = \angle EBA = 22\frac{1}{2}°$; prove that $AO^2 = 2OC^2$.

2. PQ is a chord of a circle; T is a point on the tangent at P such that PT=PQ; TQ cuts the circle at R; prove $\angle RPT = 60° \pm \frac{1}{3}\angle QPR$.

3. In Fig. 326, AB, CD, EF are parallel; AD=7″, DF=3″, CE=4″; find BC. If EF=2″, AB=3″, find CD.

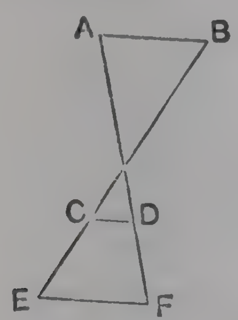

FIG. 326.

4. AB, DC are parallel sides of the trapezium ABCD; AC cuts DB at O; the line through O parallel to AB cuts AD, BC at P, Q; prove PO=OQ.

62.

1. In $\triangle ABC$, $AB=AC$ and $\angle BAC=120°$; the perpendicular bisector of AB cuts BC at X; prove $BC=3BX$.

2. AOB, COD are two perpendicular chords of a circle; prove that arc AC + arc BD equals half the circumference.

3. A light is placed 4′ in front of a circular hole 3″ in diameter in a partition; find the diameter of the illuminated part of a wall 5′ behind the partition and parallel to it.

4. ABC is a triangle inscribed in a circle; $AB=AC$; AP is a chord cutting BC at Q; prove $AP \cdot AQ=AB^2$.

63.

1. In $\triangle ABC$, $\angle BAC=90°$, $\angle ABC=45°$; AB is produced to D so that $AD \cdot DB=AB^2$; prove that the perpendicular bisector of CD bisects AB.

2. ABCD is a cyclic quadrilateral; AC cuts BD at O; if CD touches the circle OAD, prove that CB touches the circle OAB.

3. ABCDEF is a straight line; $AB : BC : CD : DE : EF = 2 : 3 : 7 : 4 : 5$; find the ratios $\frac{AD}{EF}$ and $\frac{BE}{AF}$.

4. ABCD is a parallelogram; a line through A cuts BD, BC, CD at E, F, G; prove $\frac{AE}{EF}=\frac{AG}{AF}$.

64.

1. AB is a diameter of a circle APB; the tangent at A meets BP at Q; prove that the tangent at P bisects AQ.

2. PAQ, PBQ, PCQ are three equal angles on the same side of PQ; the bisectors of $\angle$s PAQ, PBQ meet at H; prove that CH bisects $\angle PCQ$.

3. Two triangles are equiangular: the sides of one are 3 cm., 5 cm., 7 cm.; the perimeter of the other is $2\frac{1}{2}$ feet; find its sides.

4. Two lines OAB, OCD cut a circle at A, B, C, D; H, K are points on OB, OD such that $OH=OC$, $OK=OA$; prove that HK is parallel to BD.

65.

1. C is the mid-point of AB; P is any point on CB; prove that $AP^2-PB^2=2AB \cdot CP$.

2. A circular cylinder of height 6″ is cut from a sphere of radius 4″; find its greatest volume.

3. Show that the triangle whose vertices are (2, 1), (5, 1), (4, 2) is similar to the triangle whose vertices are (1, 1), (7, 1), (5, 3).

4. Two circles intersect at A, B; the tangents at A meet the circles at C, D; prove $\frac{BC}{BA}=\frac{BA}{BD}$.

66.

1. ABCD is a quadrilateral; AP is drawn equal and parallel to BD; prove $\triangle$APC=quad. ABCD.

2. A circular cone is made from a sector of a circle of radius 6″ and angle 240°; find its height.

3. A straight rod AB, 3′ 9″ long, is fixed under water with A 2′ 6″ and B 9″ below the surface; what is the depth of a point C on the rod where AC=1′?

4. ABCD is a straight line; O is a point outside it; a line through B parallel to OD cuts OA, OC at P, Q; if PB=BQ, prove $\frac{AB}{BC}=\frac{AD}{CD}$.

67.

1. In Fig. 327, OA, AB are two rods hinged together at A; the end O is fixed, and AO can turn freely about it; the end B is constrained to slide in a fixed groove OC.

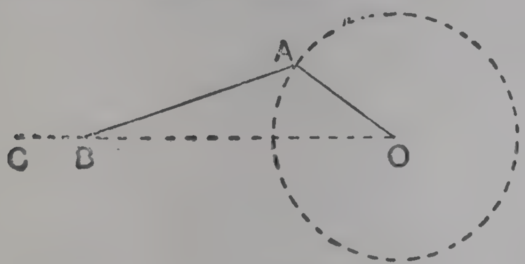

Fig. 327.

OA=3′, AB=4′; find the greatest length of the groove which B can travel over, and calculate the distance of B from O when AB makes the largest possible angle with OC.

2. ABC is a triangle inscribed in a circle; P, Q, R are the mid-points of the arcs BC, CA, AB; prove AP is perpendicular to QR.

3. AOXB, COYD are two straight lines; AC, XY, BD are parallel lines cutting them; AX=7, XB=3, AC=2, BD=4; find XY.

4. P is any point on the common chord of two circles, centres A, B; HPK and XPY are chords of the two circles perpendicular to PA, PB respectively; prove HK=XY.

68.

1. ABC is a triangle inscribed in a circle; the internal and external bisectors of $\angle$BAC cut BC at P, Q; prove that the tangent at A bisects PQ.

2. A circle of radius 4 cm. touches two perpendicular lines; calculate the radius of the circle touching this circle and the two lines.

3. ABCD is a rectangle; AB=8″, BC=5″; P is a point inside it whose distances from AD, AB are 2″, 1″; DP is produced to meet AB at E; CE cuts AD at F; calculate EB, AF.

4. Two lines OAB, OCD meet a circle at A, B, C, D; prove that $\frac{OA \cdot OD}{OB \cdot OC}=\frac{AD^2}{BC^2}$.

69.

1. ABC is an equilateral triangle; P is any point on BC; AC is produced to Q so that CQ=BP; prove AP=PQ.

2. AB is a diameter of a circle APB; AH, BK are the perpendiculars from A, B to the tangent at P; prove that AH+BK=AB.

3. A chord AB of a circle ABT is produced to O; OT is a tangent; OA=6″, OT=4″, AT=3″; find BT.

4. AB, DC are parallel sides of the trapezium ABCD; AC cuts BD at E; DA, CB are produced to meet at F; EF cuts AB, DC at P, Q; prove $\frac{QE}{EP}=\frac{QF}{PF}$.

70.

1. A brick rests on the ground and an equal brick is propped up against it as in Fig. 328. The bricks are 4″ by 2″. Calculate

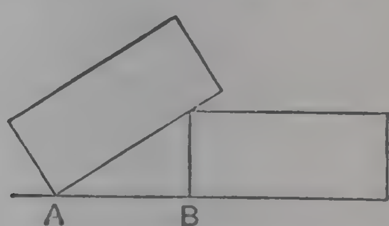

FIG. 328.

the height of each corner of the second brick above the ground, if AB=$1\frac{1}{2}$″.

2. Prove that the area of a square inscribed in a given semicircle is $\frac{2}{5}$ of the area of the square inscribed in the whole circle.

3. The bisector of $\angle BAC$ cuts BC at D; the line through D perpendicular to DA cuts AB, AC at Y, Z; prove $\frac{BY}{CZ}=\frac{BD}{DC}$.

4. A chord AD is parallel to a diameter BC of a circle; the tangent at C meets AD at E; prove $BC \cdot AE=BD^2$.

71.

1. A is a fixed point on a given circle; a variable chord AP is produced to Q so that PQ is of constant length; QR is drawn perpendicular to AQ; prove that QR touches a fixed circle.

2. Four equal circular cylinders, diameter 4″, length 5″, are packed in a rectangular box; what is the least amount of unoccupied space in the box?

3. A rectangular sheet of paper ABCD is folded so that B falls on CD and the crease passes through A; AB=10″, BC=6″; find the distance of the new position of B from C. If the crease meets BC at Q, find CQ.

4. ABCD is a parallelogram; a line through A cuts BD, CD, BC in P, Q, R; prove $\frac{PQ}{PR}=\frac{PD^2}{PB^2}$.

72.

1. In Fig. 329, ABCD is a rectangle; BP=2CQ; AD=2AB=6″. The area of APQD is 10 sq. in.; find BP.

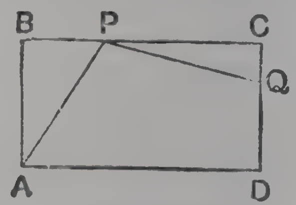

Fig. 329.

2. ABC is a triangle inscribed in a circle; the tangents at B, C meet at T; a line through T parallel to the tangent at A meets AB, AC produced at D, E; prove DT=TE.

3. A line HK parallel to BC cuts AB, AC at H, K; the distance between HK and BC is 5 cm.; the areas of AHK and HKCB are 9 sq. cms., 40 sq. cms.; find HK.

4. In $\triangle ABC$, I is the in-centre and I_1 is the ex-centre corresponding to BC; prove $AI \cdot AI_1=AB \cdot AC$.

APPENDIX I.

TREATMENT OF FUNDAMENTAL THEOREMS.

A formal treatment of the theorems regarded as intuitive in Stage B is given below.

Definition.

If C is any point on the straight line AB, and if a line CD is

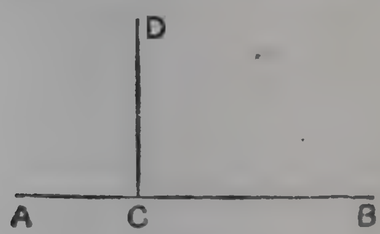

FIG. 330.

drawn so that the angles ACD, BCD are equal, each is called a *right angle.*

Therefore if C is any point on the straight line AB, the angle ACB is equal to two right angles, or 180°.

THEOREM 1.

(1) If one straight line stands on another straight line, the sum of the two adjacent angles is two right angles.

(2) If at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines are in the same straight line.

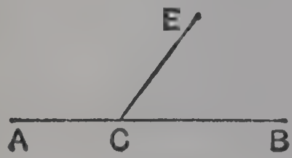

FIG. 331.

(1) *Given* CE meets AB at C.

To prove $\angle ACE + \angle BCE = 180°$.

$$\angle ACE + \angle BCE = \angle ACB$$
$$= 180°, \text{ since ACB is a st. line.}$$

Q.E.D.

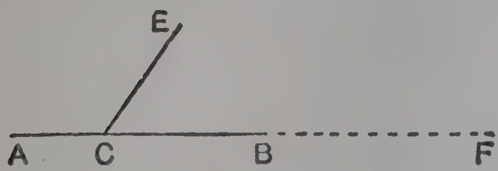

FIG. 332.

(2) *Given* $\angle ACE + \angle BCE = 180°$.

To prove ACB is a straight line.

Produce AC to F.

$\therefore\ \angle ACE + \angle FCE = 180°$, since ACF is a st. line.

But $\angle ACE + \angle BCE = 180°$, given.

$\therefore\ \angle ACE + \angle FCE = \angle ACE + \angle BCE$.

$\therefore\ \angle FCE = \angle BCE$.

$\therefore$ CB falls along CF.

But ACF is a st. line; $\therefore$ ACB is a st. line.

Q.E D.

THEOREM 2.

If two straight lines intersect, the vertically opposite angles are equal.

To prove that $x=y$ and $\alpha=\beta$.

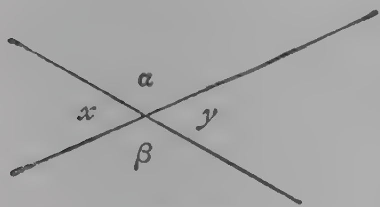

FIG. 333.

$x+\alpha=180°$, adjacent angles.

$\alpha+y=180°$, adjacent angles.

$\therefore\ x+\alpha=\alpha+y.$

$\therefore\ x=y.$

Similarly $\alpha=\beta$. Q.E.D.

THEOREM 8.

If two triangles have two sides of one equal respectively to two sides of the other, and if the included angles are equal, then the triangles are congruent.

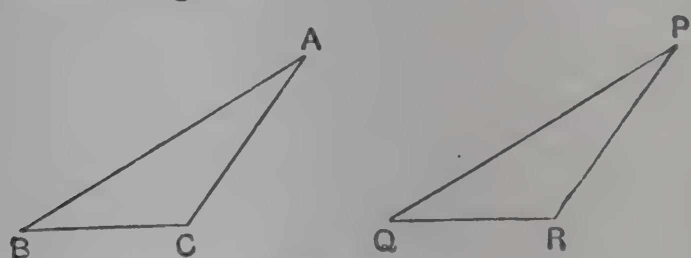

FIG. 334.

Given AB = PQ, AC = PR, ∠BAC = ∠QPR.

To prove △ABC ≡ △PQR.

Apply the triangle ABC to the triangle PQR, so that A falls on P and the line AB along the line PQ;

Since AB = PQ, ∴ B falls on Q.

Also since AB falls along PQ and ∠BAC = ∠QPR, ∴ AC falls along PR.

But AC = PR, ∴ C falls on R.

∴ the triangle ABC coincides with the triangle PQR.

∴ △ABC ≡ △PQQ. Q.E.D.

THEOREM.

If one side of a triangle is produced, the exterior angle is greater than either of the interior opposite angles.

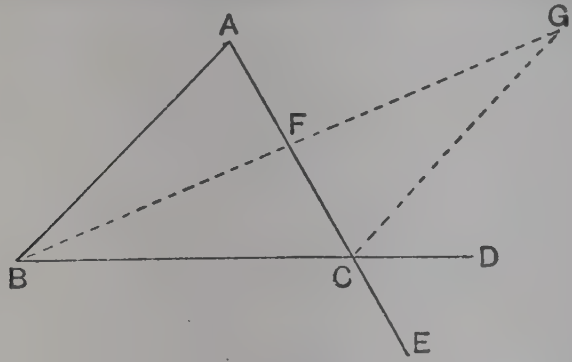

FIG. 335.

BC is produced to D.

To prove $\angle ACD > \angle ABC$ and $\angle ACD > \angle BAC$.

Let F be the middle point of AC. Join BF and produce it to G, so that $BF = FG$. Join CG.

In the triangles AFB, CFG.

$$AF = FC \text{ and } BF = FG, \text{ constr.}$$

$$\angle AFB = \angle CFG, \text{ vert. opp.}$$

$$\therefore \triangle AFB \equiv \triangle CFG \text{ (2 sides, inc. angle).}$$

$$\therefore \angle BAF = \angle GCF.$$

$$\text{But } \angle DCA > \text{its part } \angle GCF.$$

$$\therefore \angle DCA > \angle BAF \text{ or } \angle BAC.$$

Similarly, if BC is bisected and if AC is produced to E, it can be proved that $\angle BCE > \angle ABC$.

$$\text{But } \angle ACD = \angle BCE, \text{ vert. opp.}$$

$$\therefore \angle ACD > \angle ABC.$$

Q.E.D.

Definition.

Straight lines which *lie in the same plane* and which never meet, however far they are produced either way, are called *parallel* straight lines.

Playfair's Axiom.

Through a given point, one and only one straight line can be drawn parallel to a given straight line.

THEOREM 3.

If one straight line cuts two other straight lines such that
either (1) the alternate angles are equal,
or (2) the corresponding angles are equal,
or (3) the interior angles on the same side of the cutting line are supplementary,
then the two straight lines are parallel.

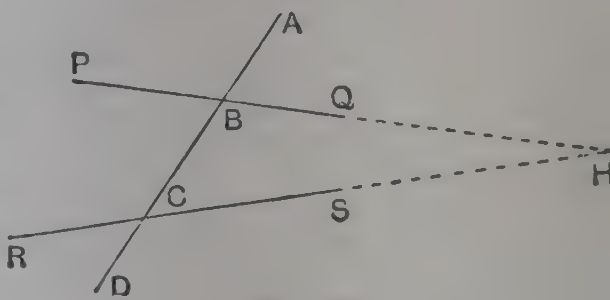

FIG. 336.

ABCD cuts PQ, RS at B, C.

(1) *Given* $\angle PBC = \angle BCS$.

To prove PQ is parallel to RS.

If PQ, RS are not parallel, they will meet when produced, at H, say.

Since BCH is a triangle,

$$\text{ext. } \angle PBC > \text{int. } \angle BCH,$$

which is contrary to hypothesis.

$\therefore$ PQ cannot meet RS and is $\therefore$ parallel to it.

Q.E.D

(2) *Given* $\angle ABQ = \angle BCS$.

To prove PQ is parallel to RS.

$\angle ABQ = \angle PBC$, vert. opp.

But $\angle ABQ = \angle BCS$, given.

$\therefore$ $\angle PBC = \angle BCS$.

$\therefore$ by (1), PQ is parallel to RS.

(3) *Given* $\angle QBC + \angle SCB = 180°$.

To prove PQ is parallel to RS.

$\angle QBC + \angle PBC = 180°$, adj. angles, QBP a st. line.

But $\angle QBC + \angle SCB = 180°$, given.

$\therefore$ $\angle QBC + \angle PBC = \angle QBC + \angle SCB$.

$\therefore$ $\angle PBC = \angle SCB$.

$\therefore$ by (1), PQ is parallel to RS. Q.E.D.

THEOREM 4.

If a straight line cuts two parallel straight lines,

Then (1) the alternate angles are equal;

(2) the corresponding angles are equal;

(3) the interior angles on the same side of the cutting line are supplementary.

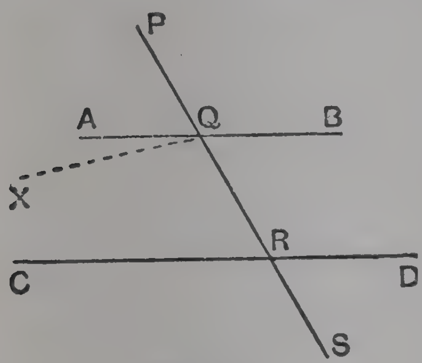

FIG. 337.

AB, CD are two parallel st. lines; the line PS cuts them at Q, R.

To prove (1) $\angle AQR = \angle QRD$.

(2) $\angle PQB = \angle QRD$.

(3) $\angle BQR + \angle QRD = 180°$.

(1) If $\angle AQR$ is not equal to $\angle QRD$, let the angle XQR be equal to $\angle QRD$.

But these are alternate angles.

$\therefore$ QX is parallel to RD,

$\therefore$ two intersecting lines QX, QA are both parallel to RD, which is impossible by Playfair's Axiom.

$\therefore$ $\angle AQR$ cannot be unequal to $\angle QRD$.

$\therefore$ $\angle AQR = \angle QRD$.

(2) $\angle PQB = \angle AQR$, vert. opp.

But $\angle AQR = \angle QRD$, alt. angles, by (1).

$\therefore$ $\angle PQB = \angle QRD$.

(3) $\angle BQR + \angle AQR = 180°$, adj. angles, BQA a st. line.

But $\angle AQR = \angle QRD$, alt. angles, by (1).

$\therefore$ $\angle BQR + \angle QRD = 180°$. Q.E.D.

THEOREM 5.

Straight lines which are parallel to the same straight line are parallel to one another.

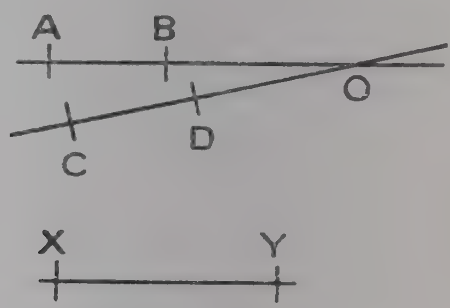

FIG. 338.

Given that AB and CD are each parallel to XY.

To prove that AB is parallel to CD.

If possible let AB cut CD (produced if necessary) at O.

Then through O there are two straight lines OA, OC, both of which are parallel to XY.

But this is impossible by Playfair's Axiom.

∴ AB cannot cut CD and must therefore be parallel to it.

Q.E.D.

THEOREM 9.

Two triangles are congruent if two angles and a side of one are respectively equal to two angles and the corresponding side of the other.

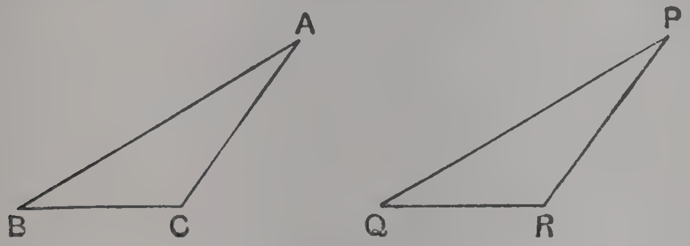

FIG. 339.

Given either that $BC = QR.$

$\angle ABC = \angle PQR.$

$\angle ACB = \angle PRQ.$

or that $BC = QR.$

$\angle ABC = \angle PQR.$

$\angle BAC = \angle QPR.$

To prove $\triangle ABC \equiv \triangle PRQ.$

The sum of the three angles of any triangle is 180°.

$\therefore$ in each case, the remaining pair of angles is equal.

Apply the triangle ABC to the triangle PQR so that B falls on Q and BC falls along QR.

Since $BC = QR$, C falls on R.

And since BC falls on QR and $\angle ABC = \angle PQR$, $\therefore$ BA falls along QP.

And since CB falls on RQ and $\angle ACB = \angle PRQ$, $\therefore$ CA falls along RP.

$\therefore$ A falls on P.

$\therefore$ the triangle ABC coincides with the triangle PQR.

$$\therefore \triangle ABC \equiv \triangle PQR.$$

Q.E.D.

THEOREM 10.

Two triangles are congruent if the three sides of one are respectively equal to the three sides of the other.

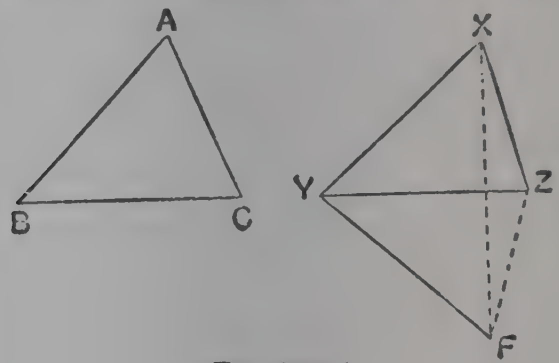

FIG. 340 (1)

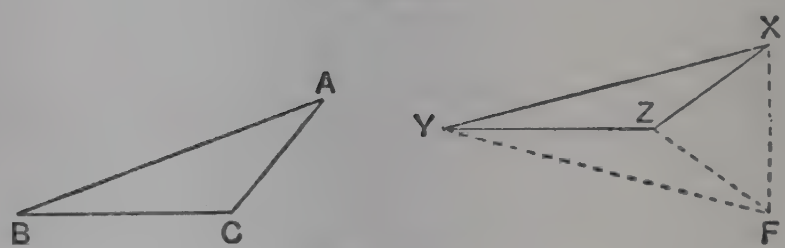

FIG. 340 (2)

Given that $AB = XY$, $BC = YZ$, $CA = ZX$.

To prove $\triangle ABC \equiv \triangle XYZ$.

Place the triangle ABC so that B falls on Y and BC along YZ;

$\therefore$ since $BC = YZ$, C falls on Z.

Let the point A fall at a point F on the opposite side of YZ to X. Join XF.

Now $YF = BA$, constr.

But $BA = YX$, given. $\therefore$ $YF = YX$.

But these are sides of the triangle YFX. $\therefore$ $\angle YXF = \angle YFX$.

Similarly, $\angle ZXF = \angle ZFX$.

$\therefore$ adding in Fig. 340 (1) or subtracting in Fig. 340 (2),

$\angle YXZ = \angle YFZ$.

But $\angle BAC = \angle YFZ$, constr. $\therefore$ $\angle BAC = \angle YXZ$.

$\therefore$ in the $\triangle$s ABC, XYZ,

$AB = XY$, given. $AC = XZ$, given.

$\angle BAC = \angle YXZ$, proved.

$\therefore$ $\angle ABC \equiv \angle XYZ$ (2 sides, inc. angle). Q.E.D.

THEOREM 15.

Any two sides of a triangle are together greater than the third side.

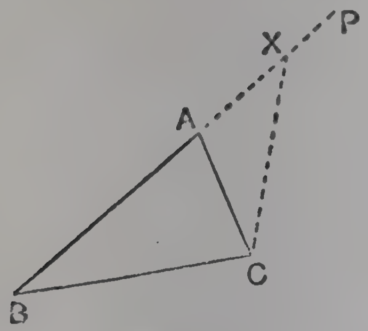

FIG. 341.

Given the triangle ABC.

To prove $BA + AC > BC$.

Produce BA to P and cut off AX equal to AC. Join CX.

Since $AX = AC$ $\angle ACX = \angle AXC$.

But $\angle BCX > \angle ACX$.

$\therefore$ $\angle BCX > \angle AXC$.

$\therefore$ in the triangle BXC, $\angle BCX > \angle BXC$.

$\therefore$ $BX > BC$.

But $BX = BA + AX = BA + AC$.

$\therefore$ $BA + AC > BC$.

Q.E.D.

THEOREM 40.

In equal circles (or in the same circle), if two arcs subtend equal angles at the centres or at the circumferences, they are equal.

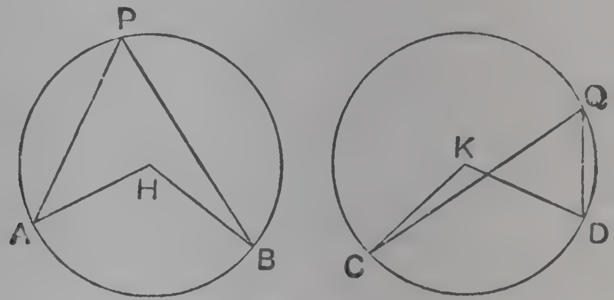

FIG. 342.

Given two equal circles, ABP, CQD, centres H, K.

(1) *Given* that $\angle AHB = \angle CKD$.

To prove that arc AB = arc CD.

Apply the circle AB to the circle CD so that the centre H falls on the centre K and HA along KC.

Since the circles are equal, A falls on C and the circumferences coincide.

Since $\angle AHB = \angle CKD$, HB falls on KD, and B falls on D.

$\therefore$ the arcs AB, CD coincide.

$\therefore$ arc AB = arc CD.

(2) *Given* that $\angle APB = \angle CQD$.

To prove that arc AB = arc CD.

Now $\angle AHB = 2\angle APB$, $\angle$ at centre = twice $\angle$ at ○ce,

and $\angle CKD = 2\angle CQD$.

But $\angle APB = \angle CQD$, given.

$\therefore$ $\angle AHB = \angle CKD$.

$\therefore$ arc AB = arc CD.

Q.E.D.

THEOREM 41.

In equal circles (or in the same circle), if two arcs are equal, they subtend equal angles at the centres and at the circumferences.

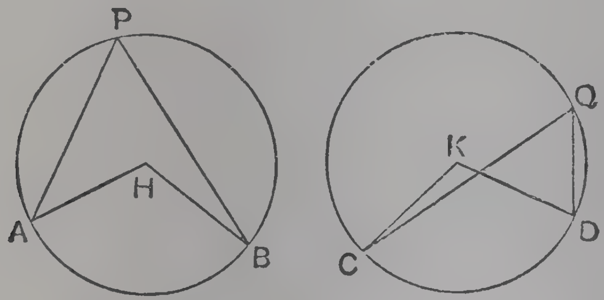

FIG. 343.

Given two equal circles ABP, CDQ, centres H, K, and two equal arcs AB, CD.

To prove (1) $\angle AHB = \angle CKD$.

(2) $\angle APB = \angle CQD$.

(1) Apply the circle AB to the circle CD so that the centre H falls on the centre K and HA along KC.

Since the circles are equal, A falls on C and the circumferences coincide.

But arc AB = arc CD, $\therefore$ B falls on D and HB on KD.

$\therefore$ $\angle AHB$ coincides with $\angle CKD$.

$\therefore$ $\angle AHB = \angle CKD$.

Q.E.D.

(2) Now $\angle APB = \frac{1}{2} \angle AHB$. $\angle$ at $\bigcirc$ce $= \frac{1}{2} \angle$ at centre.

$\angle CQD = \frac{1}{2} \angle CKD$.

But $\angle AHB = \angle CKD$, just proved.

$\therefore$ $\angle APB = \angle CQD$.

Q.E.D.

THEOREM.

If two triangles have equal heights, the ratio of their area is equal to the ratio of their bases.

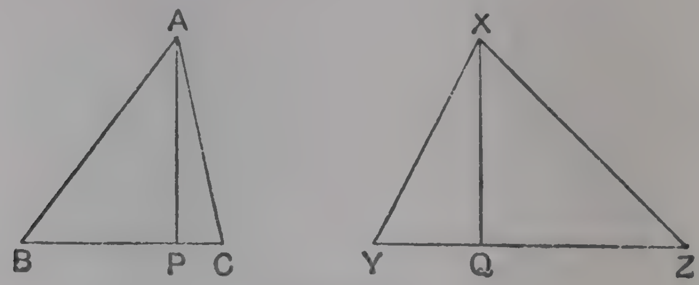

FIG. 344.

Given two triangles ABC, XYZ having equal heights AP, XQ.

To prove $\frac{\triangle ABC}{\triangle XYZ}=\frac{BC}{YZ}$.

The area of a triangle $=\frac{1}{2}$ height × base.

$$\therefore \triangle ABC=\tfrac{1}{2}AP \,.\, BC,$$

$$\text{and } \triangle XYZ=\tfrac{1}{2}XQ \,.\, YZ,$$

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{\frac{1}{2}AP \,.\, BC}{\frac{1}{2}XQ \,.\, YZ}.$$

But $AP=XQ$, given,

$$\therefore \frac{\triangle ABC}{\triangle XYZ}=\frac{BC}{YZ}.$$

Q.E.D.

THEOREMS 52 and 53.

(1) If a straight line is drawn parallel to one side of a triangle, it divides the other sides (produced if necessary) proportionally.

(2) If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

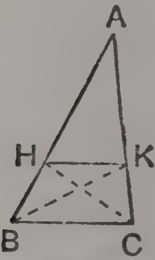

FIG. 345 (1)

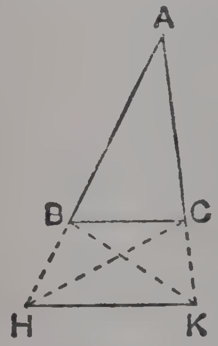

FIG. 345 (2)

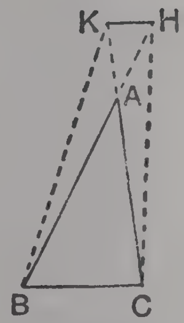

FIG. 345 (3)

(1) *Given* a line parallel to BC cuts AB, AC (produced if necessary) at H, K.

To prove $\frac{AH}{HB}=\frac{AK}{KC}$.

Join BK, CH.

The triangles KHA, KHB have a common altitude from K to AB.

$$\therefore \frac{\triangle KHA}{\triangle KHB}=\frac{AH}{HB}.$$

The triangles HKA, HKC have a common altitude from H to AC.

$$\therefore \frac{\triangle HKA}{\triangle HKC}=\frac{AK}{KC}.$$

But $\triangle KHB$, $\triangle KHC$ are equal in area, being on the same base HK and between the same parallels HK, BC.

$$\therefore \frac{AH}{HB}=\frac{AK}{KC}.$$

Q.E.D.

(2) *Given* a line HK cutting AB, AC at H, K such that $\frac{AH}{HB}=\frac{AK}{KC}$.

To prove HK is parallel to BC.

The triangles KHA, KHB have a common altitude from K to AB.

$$\therefore \frac{\triangle KHA}{\triangle KHB}=\frac{AH}{HB}.$$

The triangles HKA, HKC have a common altitude from H to AC.

$$\therefore \frac{\triangle HKA}{\triangle HKC}=\frac{AK}{KC}.$$

But $\frac{AH}{HB}=\frac{AK}{KC}$, given.

$$\therefore \frac{\triangle KHA}{\triangle KHB}=\frac{\triangle HKA}{\triangle HKC}.$$

$$\therefore \triangle KHB=\triangle HKC.$$

But these triangles are on the same base HK and on the same side of it.

$\therefore$ HK is parallel to BC.

Q.E.D.

APPENDIX II.

TREATMENT OF LIMITS.

It is easier to write out in an examination a correct proof of a tangent property by the method of Euclid than by the use of limits. But the method of limits is valuable because it presents a new aspect of geometry and prepares the way for such calculus and mechanics as now forms part of most non-specialist school courses.

Theorem 42 may be deduced as the limit of the theorem that the line joining the centre of a circle to the middle point of a chord is at right angles to the chord.

Theorem 43 may be deduced as the limit of the theorem that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

Under favourable conditions it may, however, be possible to analyse more closely the nature of limit proofs, and in this case the following treatment is suggested. It is intended as a basis for oral discussion and not for reproduction in examinations. It offers a simple illustration of the "Method of Exhaustion" which historically was the fore-runner of the technique of the Calculus.

The Idea of a Limit.

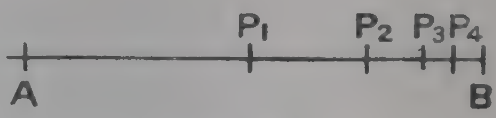

FIG. 346.

Draw a straight line AB, one decimetre long. Bisect AB at P_1; bisect P_1B at P_2; bisect P_2B at P_3; etc.

Then $AP_1 = \frac{1}{2}$ dm.; $AP_2 = (\frac{1}{2} + \frac{1}{4})$ dm.; $AP_3 = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8})$ dm.; etc.

We can repeat the bisection process as often as we like; the successive distances from A of the points "P" measured in dm. are
$$\tfrac{1}{2};\ \tfrac{1}{2} + \tfrac{1}{4};\ \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8};\ \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16};\ \ldots.$$

However often the process is repeated we never arrive at B.

Therefore, however many terms in the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ are taken and summed the result is always less than 1.

But by repeating the process sufficiently often we can obtain a point P as near B as we like, because $P_1B = \frac{1}{2}$, $P_2B = \frac{1}{4}$, $P_3B = \frac{1}{8}, \ldots$.

And so, by taking a sufficient number of terms in the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ we can obtain a sum as near 1 as we like; and, further, any greater number of terms will give a sum still nearer to 1.

We therefore say that the point P_n tends to B as its limiting position, or that the **limit** of P_n is B when n increases indefinitely. The limit is, however, never attained, because P_n never coincides with B.

We also say that the **limiting sum** of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is 1, or that the sum of the series tends to 1 when the number of terms increases indefinitely. But the limit 1 is never attained because, however many terms are taken, their sum is always less than 1.

The Tangent as a Limiting Chord.

A is a given point on a circle. Any line AQ through A cuts the circle at P.

Bisect arc AP at P_1; bisect arc AP_1 at P_2; bisect arc AP_2 at P_3; etc.

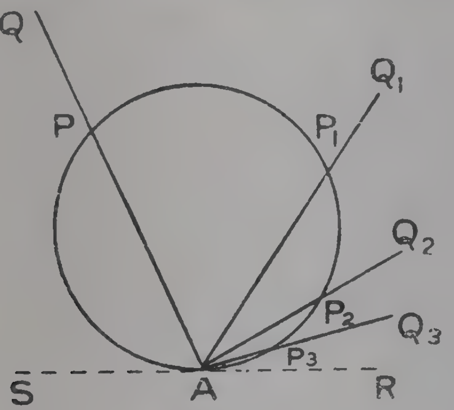

FIG. 347.

We can repeat this bisection process as often as we like, and thus obtain a succession of lines AP_1Q_1, AP_2Q_2, AP_3Q_3, ... which cut off from the circle arcs of continually decreasing lengths. However often we repeat the process we cannot obtain a line cutting off an arc of zero length; but by repeating it sufficiently often we can obtain a line which cuts off an arc as short as we please, and all further lines obtained will cut off still shorter arcs.

The limiting position AR of this series of lines is called **the tangent at** A, which is, therefore, the limiting position of a line through A cutting off an arc whose length decreases without limit.

If the process of repeated bisection is performed on the other side of the chord AP, we obtain the limiting position AS which is equally by our definition the tangent at A. It is, therefore, necessary to prove that AR and AS are in one straight line.

The fact that the limiting position of AR does not depend on the position of the first line AP will be proved by showing that there is only one tangent at A, *i.e.* that there is only one line through A which cuts off an arc of zero length.

THEOREM 42.

A tangent at any point of a circle is perpendicular to the radius through that point at its extremity.

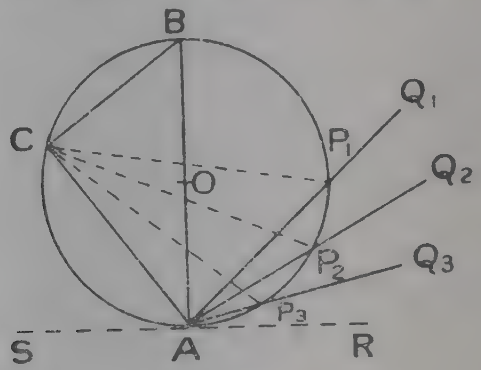

FIG. 348.

Given that AR is a tangent at A and AOB is a diameter and O the centre.

To prove that $\angle BAR = 90°$.

Bisect arc AB at P_1, bisect arc AP_1 at P_2, bisect arc AP_2 at P_3, etc.

Then the lines AP_1Q_1, AP_2Q_2, AP_3Q_3, ... form a series of lines whose limiting position is AR.

Method 1.

Take any point C on the circle on the side of AB remote from R.

Join CP_1, CP_2, CP_3, ... , these lines form a series of lines whose limiting position is CA.

We therefore have two series of angles (i) $B\hat{A}Q_1, B\hat{A}Q_2, B\hat{A}Q_3, \ldots,$ and (ii) $B\hat{C}P_1$, $B\hat{C}P_2$, $B\hat{C}P_3$, ..., which tend to exhaust the angles BAR, BCA respectively.

But $B\hat{A}Q_1 = B\hat{C}P_1$, $B\hat{A}Q_2 = B\hat{C}P_2$, etc., standing on the same arc

$$\therefore \text{ in the limit } \angle BAR = \angle BCA.$$

But $\angle BCA = 90°$, $\angle$ in semi-circle. $\therefore$ $\angle BAR = 90°$.

Q.E.D

Method 2.

Join $OP_1, OP_2, OP_3, \dots$.

Then $B\hat{A}P_1 = \frac{1}{2}B\hat{O}P_1 = \frac{1}{2}$ right angle, since arc BP_1 = arc P_1A,

$P_1\hat{A}P_2 = \frac{1}{2}B\hat{A}P_1 = \frac{1}{4}$ right angle, since arc $P_1P_2 = \frac{1}{2}$ arc BP_1,

$P_2\hat{A}P_3 = \frac{1}{2}P_1\hat{A}P_2 = \frac{1}{8}$ right angle, since arc $P_2P_3 = \frac{1}{2}$ arc P_1P_2,

and so on.

$\therefore$ $B\hat{A}P_n = (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots)$ right angles, n terms.

But by taking n sufficiently large, we can make the sum of these terms as near 1 as we like.

But AR is the limiting position of AP_n when n increases indefinitely.

$\therefore$ in the limit, we have $\angle BAR = 1$ right angle.

Q.E.D.

Corollary 1. *There is not more than one tangent at any given point on a circle, and every other line through that point cuts the circle at a second point.*

The process of repeated bisection on the other side of AB gives a limiting position AS as a tangent at A.

By the same argument it may be proved that $S\hat{A}B = 90°$.

$\therefore$ $S\hat{A}B + B\hat{A}R = 180°$.

$\therefore$ SAR is a straight line.

Further, any other line AX through A must lie either in the angle BAR or the angle BAS ; suppose it lies in the angle BAR.

Since $B\hat{A}X < 90°$, a line AP_n can be found such that

$$B\hat{A}X < B\hat{A}P_n.$$

$\therefore$ AX lies between AB and AP_n.

$\therefore$ AX must cut the circle at some point on the minor arc BP_n.

Corollary 2. The line drawn perpendicular to the radius through its extremity is a tangent to the circle.

The bisection process shows that a tangent at any point of a circle exists, and the theorem shows that the tangent coincides with the perpendicular to the radius at its extremity.

THEOREM 43.

The angles which a tangent to a circle makes with any chord drawn through the point of contact are equal to the angles in the alternate segments.

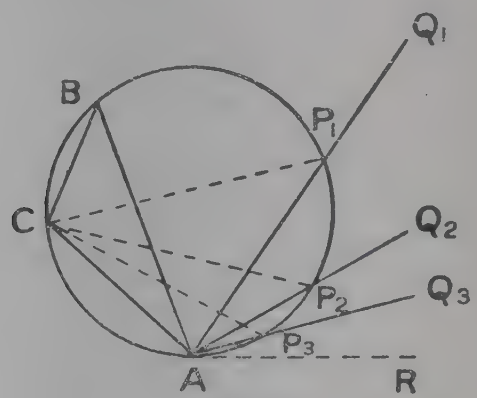

FIG. 349.

Given AR is the tangent at A, AB is any chord.

To prove $B\hat{A}R = A\hat{C}B$ in alternate segment.

Bisect arc AB at P_1, P_1 and C being on opposite sides of AB.

Bisect arc AP_1 at P_2; bisect arc AP_2 at P_3; and so on.

Then the lines AP_1Q_1, AP_2Q_2, AP_3Q_3, ... form a series of lines whose limiting position is AR.

Also the lines CP_1, CP_2, CP_3, ... form a series of lines whose limiting position is CA.

We therefore have two series of angles (i) $B\hat{A}Q_1$, $B\hat{A}Q_2$, $B\hat{A}Q_3$, ..., and (ii) $B\hat{C}P_1$, $B\hat{C}P_2$, BCP_3, ..., which tend to exhaust the angles BAR, BCA respectively.

But $B\hat{A}Q_1 = B\hat{C}P_1$, $B\hat{A}Q_2 = B\hat{C}P_2$, etc., standing on the same arc.

$\therefore$ in the limit, $B\hat{A}R = B\hat{C}A$. Q.E.D.

APPENDIX III.

THE TRIANGLE—CONCURRENCY PROPERTIES.

THEOREM 60.

The perpendicular bisectors of the three sides of a triangle are concurrent (*i.e.* meet in a point).

Given that the perpendicular bisectors OY, OZ of AC, AB meet at O.

To prove the perpendicular bisector of BC passes through O.

Bisect BC at X, join OX; also join OA, OB, OC.

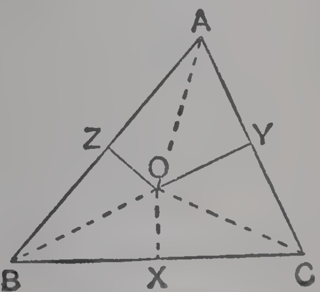

FIG. 350.

In the $\triangle$s OZA, OZB,

BZ = ZA, given.

OZ is common.

$\angle$BZO = $\angle$AZO, given rt. $\angle$s.

$\therefore$ $\triangle$OZA $\equiv$ $\triangle$OZB (2 sides, inc. $\angle$).

$\therefore$ OA = OB.

Similarly from the $\triangle$s OYA, OYC, it can be proved that

OA = OC, $\therefore$ OB = OC.

In the $\triangle$s OXB, OXC, OB = OC, proved.

XB = XC, constr.

OX is common.

$\therefore$ $\triangle$OXB $\equiv$ $\triangle$OXC (3 sides). $\therefore$ $\angle$OXB = $\angle$OXC.

But these are adjacent angles, $\therefore$ each is a rt. $\angle$.

$\therefore$ OX is the perpendicular bisector of BC. Q.E.D.

O is the centre of the circumcircle of the triangle ABC and is called the **circumcentre**.

THEOREM 61.

The internal bisectors of the three angles of a triangle are concurrent.

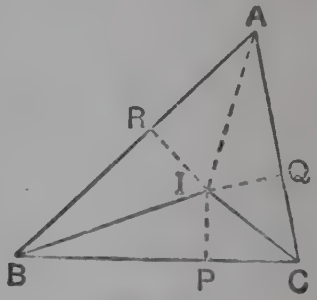

FIG. 351.

Given that the internal bisectors IB, IC of the angles ABC, ACB meet at I.

To prove that IA bisects the angle BAC.

Join IA. Draw IP, IQ, IR perpendicular to BC, CA, AB.

In the $\triangle$s IBP, IBR, $\angle$IBP = $\angle$IBR, given.

$\angle$IPB = $\angle$IRB, constr. rt. $\angle$s.

IB is common.

$\therefore$ $\triangle$IBP $\equiv$ $\triangle$IBR (2 angles, corr. side).

$\therefore$ IP = IR.

Similarly from the $\triangle$s ICP, ICQ it may be proved that

IP = IQ,

$\therefore$ IQ = IR.

In the *right-angled* triangles IAQ, IAR,

IQ = IR, proved.

IA is the common hypotenuse.

$\therefore$ $\triangle$IAQ $\equiv$ $\triangle$IAR (rt. angle, hyp., side).

$\therefore$ $\angle$IAQ = $\angle$IAR.

$\therefore$ IA bisects the angle BAC. Q.E.D.

I is the centre of the circle inscribed in the triangle ABC (*i.e.* the in-circle of $\triangle$ABC) and is called the **in-centre.** The external bisectors of the angles ABC, ACB meet at a point I_1, which is the centre of the circle which touches AB produced, AC produced, BC; this circle is said to be escribed to BC and I_1 is called an **ex-centre** (see p. 184).

THEOREM 62.

The three altitudes of a triangle (*i.e.* the lines drawn from the vertices perpendicular to the opposite sides) are concurrent.

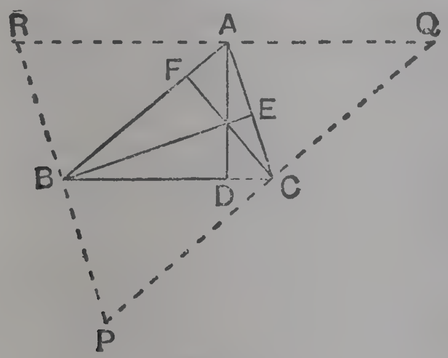

FIG. 352.

Given AD, BE, CF are the altitudes of the triangle ABC.

To prove AD, BE, CF are concurrent.

Through A, B, C draw lines parallel to BC, CA, AB to form the triangle PQR.

Since BC is || to AR and AC is || to BR,

BCAR is a parallelogram.

$\therefore$ BC = AR.

Similarly, since BCQA is a parallelogram, BC = AQ,

$\therefore$ AR = AQ.

Since AD is perpendicular to BC, and since QR, BC are parallel,

$\therefore$ AD is perpendicular to QR.

But AR = AQ, $\therefore$ AD is the perpendicular bisector of QR.

Similarly, BE and CF are the perpendicular bisectors of PR, PQ.

But the perpendicular bisectors of the sides of the triangle PQR are concurrent.

$\therefore$ AD, BE, CF are concurrent.

Q.E.D.

The point at which the altitudes concur is called the **orthocentre** of the triangle ABC. The triangle DEF is called the **pedal triangle** of $\triangle$ABC.

THEOREM 63.

(1) The three medians of a triangle (*i.e.* the lines joining each vertex to the middle point of the opposite side) are concurrrent.

(2) The point at which the medians intersect is one-third of the way up each median (measured towards the vertex).

(1) *Given* the medians BE, CF of the triangle ABC, intersect at G.

To prove that AG, when produced, bisects BC.

Join AG and produce it to H, so that AG = GH.

Let AH cut BC at D; join HB, HC.

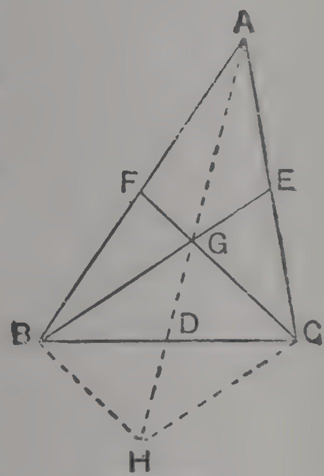

FIG. 353.

Since AF = FB and AG = GH,

FG is parallel to BH.

Since AE = EC and AG = GH,

EG is parallel to CH.

Since FGC and EGB are parallel to BH and CH,

BGCH is a parallelogram;

∴ the diagonals BC, GH bisect each other;

∴ BD = DC.

Q.E.D.

(2) For the same reason, GD = DH.

∴ GH = 2GD.

But AG = GH.

∴ AG = 2GD.

∴ AD = 3GD.

or GD = $\frac{1}{3}$AD.

Q.E.D.

The point at which the medians concur is called the centroid of the triangle ABC.

EXERCISE LXV.

THE CIRCUMCIRCLE.

1. If O is the circumcentre of $\triangle$ABC and if D is the mid-point of BC, prove $\angle$BOD$=\angle$BAC.

2. The diagonals of the quadrilateral ABCD intersect at O; P, Q, R, S are the circumcentres of $\triangle$s AOB, BOC, COD, DOA; prove PQ=RS.

3. In $\triangle$ABC, $\angle$BAC$=90°$; P is the centre of the square described on BC; prove that AP bisects $\angle$BAC.

4. In $\triangle$ABC, $\angle$BAC$=90°$; prove that the perpendicular bisectors of AB and AC meet on BC.

5. ABC is a scalene triangle; prove that the perpendicular bisector of BC and the bisector of $\angle$BAC meet *outside* the triangle ABC.

6. ABCD is a parallelogram; E, F are the circumcentres of $\triangle$s ABD, BCD; prove that EBFD is a rhombus.

7. The extremities of a variable line PQ of given length lie on two fixed lines OA, OB; prove that the locus of the circumcentre of $\triangle$OPQ is a circle, centre O.

8. If the area of the triangle ABC is Δ, the radius of the circumcircle is $\dfrac{abc}{4\Delta}$; prove this for the case where $\angle$BAC$=90°$.

9. ABCD is a quadrilateral such that AB=CD; find a point O such that $\triangle$OAB $\equiv$ $\triangle$OCD.

10. AD, BE are altitudes of $\triangle$ABC; prove that the perpendicular bisectors of AD, BE, DE are concurrent.

THE IN-CIRCLE AND EX-CIRCLES.

11. In Fig. 354, if BC$=a$, CA$=b$, AB$=c$, and $s=\frac{1}{2}(a+b+c)$, prove that

(i) AY$=s-a$. (ii) AQ$=s$.
(iii) BP=XC. (iv) YQ=ZR.
(v) XP$=b \sim c$.
(vi) IX$=\dfrac{\Delta}{s}$ where Δ=area of triangle ABC.
(vii) $I_1P=\dfrac{\Delta}{s-a}$.
(viii) B, I, C, I_1 are concyclic.
(ix) AZ+BX+CY$=s$.
(x) If $\angle$BIC$=100°$, calculate $\angle$BAC.

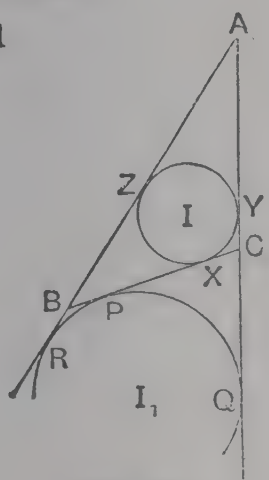

FIG. 354.

12. AB is a chord of a circle ; the tangents at A, B meet at T ; prove that the in-centre of $\triangle$TAB lies on the circle.

13. I is the in-centre and O the circumcentre of $\triangle$ABC ; prove that $\angle IAO = \frac{1}{2}(ABC \sim \angle ACB)$.

14. I is the in-centre of $\triangle$ABC ; prove that $\angle AIC = 90° + \frac{1}{2}\angle ABC$.

15. I is the in-centre and AD is an altitude of $\triangle$ABC ; prove that $\angle IAD = \frac{1}{2}(\angle ABC \sim \angle ACB)$.

16. In Fig. 354, prove that AB − AC = BX − XC.

17. The in-circle of $\triangle$ABC touches BC at X, prove that the in-circles of $\triangle$s ABX, ACX touch each other.

18. ABCD is a quadrilateral circumscribing a circle ; prove that the in-circles of $\triangle$ABC, CDA touch each other.

19. In $\triangle$ABC, $\angle BAC = 90°$; prove that the diameter of the in-circle of $\triangle$ABC equals AB + AC − BC.

20. The extremities P, Q of a variable line lie on two fixed lines AB, CD ; the bisectors of $\angle$s APQ, CQP meet at R ; find the locus of R.

21. I is the in-centre of $\triangle$ABC ; I_1 is the centre of the circle escribed to BC ; I, I_1 cuts the circumcircle of $\triangle$ABC at P ; prove that I, I_1, B, C lie on a circle, centre P.

22. I is the in-centre of $\triangle$ABC ; if the circumcircle of $\triangle$BIC cuts AB at Q, prove AQ = AC.

23. I is the in-centre of $\triangle$ABC ; AP, AQ are the perpendiculars from A to BI, CI ; prove that PQ is parallel to BC.

THE ORTHOCENTRE.

24. If AD, BE, CF are the altitudes of $\triangle$ABC and if H is its orthocentre, prove that

(i) $\angle BHF = BAC$.

(ii) $\angle BHC + \angle BAC = 180°$.

(iii) $\triangle$s AEF, ABC are equiangular.

(iv) $\triangle$s BDF, EDC are equiangular.

(v) AD bisects $\angle$FDE.

(vi) $\angle EDF = 180° - 2\angle BAC$.

(vii) H is in-centre of $\triangle$DEF.

25. Where is the orthocentre of a right-angled triangle ?

26. Q is a point inside the parallelogram ABCD such that $\angle QBC = 90° = \angle QDC$; prove that AQ is perpendicular to BD.

27. If D is the orthocentre of $\triangle ABC$, prove that A is the orthocentre of $\triangle BCD$.

28. If H is the orthocentre of $\triangle ABC$, prove that the circumcircles of $\triangle$s AHB, AHC are equal.

29. I is the in-centre and I_1, I_2, I_3 are the ex-centres of $\triangle ABC$, prove that I_1 is the orthocentre of $\triangle I I_2 I_3$.

30. In $\triangle ABC$, AB=AC, $\angle BAC = 45°$; H is the orthocentre and CHF is an altitude ; prove that BF=FH.

31. O is the circumcentre and H the orthocentre of $\triangle ABC$; prove that $\angle HBA = \angle OBC$.

32. P, Q, R are the mid-points of BC, CA, AB ; prove that the orthocentre of $\triangle PQR$ is the circumcentre of $\triangle ABC$.

33. H is the orthocentre of $\triangle ABC$; AH meets BC at D and the circumcircle of $\triangle ABC$ at P ; prove that HD=DP.

34. O is the circumcentre, I is the in-centre, H is the orthocentre of $\triangle ABC$; prove that AI bisects $\angle OAH$.

35. BE, CF are altitudes of $\triangle ABC$; O is its circumcentre ; prove that OA is perpendicular to EF.

36. H is the orthocentre and O the circumcentre of $\triangle ABC$; AK is a diameter of the circumcircle; prove that (i) BHCK is a parallelogram, (ii) CH equals twice the distance of O from AB.

37. H is the orthocentre of $\triangle ABC$; BH meets the circumcircle at K ; prove AH=AK.

38. Given the base and vertical angle of a triangle, find the locus of its orthocentre.

39. (*Nine Point Circle.*) AD, BE, CF are altitudes of $\triangle ABC$; H is its orthocentre ; X, Y, Z, P, Q, R are the mid-points of BC, CA, AB, HA, HB, HC ; prove that

(i) PZ is parallel to BE and ZX is parallel to AC.

(ii) $\angle PZX = 90°$ and $\angle PYX = 90°$.

(iii) P, Z, X, D, Y lie on a circle.

(iv) The circle through X, Y, Z passes through P, Q, R, D, E, F.

THE CENTROID.

40. X, Y, Z are the mid-points of BC, CA, AB; prove that the triangles ABC, XYZ have the same centroid.

41. ABCD is a parallelogram; P is the mid-point of AB; CP cuts BD at Q; prove that AQ bisects BC.

42. If the medians AX, BY of $\triangle$ABC meet at G, prove that $\triangle$s BGX, CGY are equal in area.

43. If G is the centroid of $\triangle$ABC and if AG=BC, prove that $\angle$BGC=90°.

44. X, Y, Z are the mid-points of BC, CA, AB; AD is an altitude of $\triangle$ABC; prove that $\angle$ZXY=$\angle$ZDY=$\angle$BAC.

45. AX, BY, CZ are the medians of $\triangle$ABC; prove that BY+CZ>AX.

46. In a tetrahedron ABCD, the plane angles at each of three corners add up to 180°; prove, by drawing the net of the tetrahedron, that its opposite edges are equal.

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ANSWERS.

PART I.

Exercise II. (p. 6.)

1. AB 4·34, 11·02 ; BC 2·59, 6·58 ; CA 3·09, 7·87 ; DE 2·58, 6·57 ; EF 3·00, 7·61 ; FD 4·86, 12·35 ; KL 2·51, 6·40 ; LM 3·13, 7·94 ; MK 3·77, 9·58.

2. AK 1·65, 4·18 ; KC 3·09, 7·87 ; BL 3·27, 8·29 ; AM 4·61, 11·7 ; FK 4·73, 12·03.

3. 2·54 cm. 4. 0·394 in. 6. 9·03 cm.

7. AL 2·10 in., CE 3·25 cm. 9. F, M, B. 14. 11, 3 in.

Exercise V. (p. 11.)

1. 1. 2. 2. 3. 3. 4. 1.
5. 1. 6. 2. 7. $1\frac{1}{2}$. 8. $\frac{1}{2}$.
9. $1\frac{1}{2}$. 10. 8. 11. 2, 3. 12. 1, 4, 8.
13. 10. 14. 20, 28. 15. 1, 1, 2, $\frac{1}{2}$. 16. S., S.W.
17. S., S.W., N.E. 18. E., S.W. 19. 3.50, 3.30 p.m.
20. 4.5, 4.25 p.m. 21. 1, $\frac{1}{3}$, 3, $1\frac{1}{2}$. 23. 200.
24. Slip. 25. 1, 4. 26. $\frac{1}{2}$. 27. $\frac{1}{2}$, $\frac{1}{2}$.
28. $1\frac{1}{2}$, $1\frac{1}{2}$. 29. $1\frac{1}{2}$, $\frac{1}{2}$. 30. 4.

Exercise VI. (p. 14.)

1. 60°. 2. 36°, 45°, 99°.
3. 45, 135, 360, 60, $22\frac{1}{2}$. 4. 3, $\frac{1}{2}$, $1\frac{1}{2}$, $1\frac{1}{3}$, $\frac{1}{3}$, $\frac{5}{6}$.
8. 34°, 146° ; 87°, 93° ; $45\frac{1}{2}$°, $133\frac{1}{2}$° ; 80°, 100° ; 50°, 130° ; 58°, 122°.
9. *e*, *b* acute ; *a*, *c* obtuse ; *d* reflex. 11. 180°.
12. 360°. 13. 77°, 103°, 283°.

Exercise VII. (p. 17.)

1. 180°. 2. 138° + 42° = 180°. 3. 50°. 5. 65°. 6. 40°.

Exercise VIII. (p. 19.)

1. 160°, 30°, 152° 15′, 87° 50′. 2. 15°, 59° 30′, 79° 12′.
3. 60°. 4. 44°, 153°, 28°, 226°. 5. 6 ; 11 ; 22.
6. 90°. 7. 135°. 8. 83° ; $112\frac{1}{2}°$; 167°.
9. 90°, 30°, 120°, 300°, 990°. 10. 20°, 30°. 11. $2^{1°}$.
12. 120°, 65°. 13. 120°. 15. 120°. 16. 72°.
17. 72°. 18. 120°. 19. $247\frac{1}{2}°$. 20. 5°.
21. 144°. 22. 74°. 23. 84° 35′ ; 11° 30′.
24. 40. 25. 110°, $149\frac{1}{2}°$. 26. 15°.
27. 46°, 180°. 28. 111°. 29. $111\frac{1}{2}°$. 30. 251°.
31. $180 - x$. 32. $90 + \frac{1}{2}x$.

Exercise X. (p. 23.)

1. 50° c ; 50° a. 5. 70°. 10. 72°, 72°, 108°.

Exercise XII. (p. 29.)

6. 36°, 45°, 99° ; $31\frac{1}{2}°$, 122°, $26\frac{1}{2}°$; 55°, 83°, 42°. 8. 4, 10 rt. $\angle$s.
9. 6 rt. $\angle$s. 10. 4. 12. (iv) 10 rt. $\angle$s.

Exercise XIII. (p. 33.)

1. 53°. 2. 74°. 3. 28°. 4. 18°.
5. 122°. 6. 93°. 7. 80°.
9. 20, $130 - x$, $180 - 5x$, 150 degrees. 10. 36.
11. 90°. 12. 80°. 13. 80. 14. Least 36°.
15. 8°. 16. 37°. 17. 86°. 19. $2x - 180°$.
20. 120°. 21. $\frac{1}{2}(x - y) + 90°$. 23. 9.
24. 20 rt. $\angle$s. 25. 162°. 27. $y = \frac{6x}{8 - x}$; $y = 6, 10, 18, 42$.
28. 6. 31. $x = c - a - b$. 32. $x = b - a - c$. 33. $x = a + b + c$.

Exercise XIV. (p. 36.)

1. 2, 8, 2 cm. 2. 4·8 cm. 11. $104\frac{1}{2}°$. 12. 4·86 in.
13. 68° 50′. 14. 113° 20′.

Exercise XV. (p. 38.)

For No. 1-10 see Answers Ex. II. No. 1 ; also A = 36°, B = 45°, C = 99°
D = $31\frac{1}{2}°$, E = 122°, F = $26\frac{1}{2}°$; K = 55°, L = 83°, M = 42°.
13. 9·49, 1·97 cm. 14. 12·65 cm. 17. 4·90 cm.

Exercise XVII. (p. 46.)

1. 94·3 ft. 2. 7140 yd. 3. 13 ft. 9 in. 4. 10 ft. 8 in.
5. 32 yd. 6. 2·77 mi. 7. S. 37° W.; 5·17 mi.
8. 7·0 mi., N. 34° W. 9. 8·42 mi., N. 12° W.
10. E. $36\frac{3}{4}°$ N. 11. 34·8 mi., N. $31\frac{1}{2}°$ W. 12. 321 yd.
13. 2·59 ft. 14. 84·0. 15. 326. 16. 31°.
17. 137. 18. 34·4.

Exercise XVIII. (p. 48.)

1. 0·466, 1, 1·73, 3·08. 2. 14°, 35°, 52°, $63\frac{1}{2}°$.
3. 4·90. 4. 15·0. 5. 5·40. 6. $41\frac{1}{4}°$.
7. $71\frac{1}{4}°$. 8. 128 ft. 9. $54\frac{1}{2}°$. 10. 353 ft.
11. E. $23\frac{3}{4}°$ N. 12. 28·8 ft. 13. $53\frac{1}{8}°$. 14. $49\frac{1}{2}°$.
15. 313 ft. 16. $28\frac{1}{2}°$. 17. 9·0 ft. 18. $37\frac{3}{4}°$, $75\frac{3}{4}°$.
19. 117 ft. 20. 30·9 ft.

Exercise XIX. (p. 50.)

1. 0·94, 0·766, 0·5. 2. 23°, 37°, 72°. 3. 3·69 cm.
4. 5·59 cm. 5. 3·28 cm. 6. 58°.
7. 6·18 ft., 18°, 19·0 ft. 8. $73\frac{3}{4}°$. 9. 28·1 ft.
10. 120, 90·3 yd. 11. 662 ft. 12. 55°.
13. 2 ft. $11\frac{1}{2}$ in. 14. E. 39° N. 15. 10·0 mi. 16. 821 ft.
17. 5·49 in. 18. 374 ft. 19. 0·79 in. 20. 219 ft.

Exercise XX. (p. 56.)

1. 35°. 2. 56°. 3. 90°, 45°; 72°, 36°.
5. 50°, 60°, 70°. 6. $x = 360 - 2y$. 7. $x = 60 \pm \frac{1}{3}y$
9. 36°. 22. $25\frac{5}{7}°$.

Exercise XXI. (p. 60.)

11. 7.

Exercise XXII. (p. 65.)

5. 68°. 6. 58°. 7. 62°. 23. $67\frac{1}{2}°$.

Exercise XXIV. (p. 75.)

12. $81\frac{3}{4}°$ or $14\frac{3}{4}°$.

Exercise XXV. (p. 80.)

1. (i) $36\frac{3}{4}^\circ$; (iii) 2·59; (iv) 2·93; (v) 4·79; (vii) 6·68; (viii) 5·66, 3·53; (xii) 11·3; (xiii) 8·49; (xiv) 8·87; (xv) $104\frac{1}{2}^\circ$.
8. 5·74. 9. 5·23. 10. $106\frac{1}{4}^\circ$. 11. $49\frac{1}{2}^\circ$. 12. $62\frac{1}{2}^\circ$.
13. 5·41. 14. 2·55. 15. 7·13, 3·63. 16. $49\frac{1}{2}^\circ$.
17. (i) 4·96; (ii) 6·76; (iii) 5·18; (iv) $63\frac{1}{4}^\circ$; (v) 3·82.
18. (i) $25\frac{1}{4}^\circ$; (ii) 8·25; (iii) 6; (iv) 6·21. 19. 8·64.
20. 3·53. 21. 4·67. 22. 7. 23. 6·09. 24. 6·16.
25. 4·26. 26. 4·96. 27. 4·62.
28. (i) 7·67; (ii) 7·10; (iii) 10·1; (iv) 4·78; (v) 7·82; (vi) 8·71; (vii) 6·64.
29. 6·22. 30. 5·34.

Exercise XXVI. (p. 86.)

19. 12, 17 in.

Exercise XXVII. (p. 89.)

6. 1·93. 7. 3·61. 8. 6·82 in. 9. $\frac{1}{31680}$. 10. 3·17.

Exercise XXVIII. (p. 90.)

3. 3·36. 5. 2·5. 6. 6·13. 7. 2·83. 24. 1·63. 25. $21\frac{3}{4}^\circ$.

Revision Papers 1-18. (pp. 93-98.)

1. 1. 30°, 15°. 2. 2″. 3. 53°.
2. 2. 0·5″. 3. 99°.
3. 1. 36°, 135°.
4. 1. 46°. 2. 36°.
5. 1. 150°.
6. 1. 108°.
7. 1. 300°. 2. Least 20°.
8. 1 72°. 2. 112°.
9. 1. 270°. 2. 110°.
10. 1. 26 rt. ∠s.
11. 1. 1° 20′ W., 3° E.
12. 1. $180 - 4x$, $\frac{1}{2}x$.
14. 1. 75°.
15. 1. $x = 540° - a - b - c$.
16. 1. $67\frac{1}{2}^\circ$.
17. 1. $z = 180° - a - b - x - y$. 2. $\frac{12}{n}$ rt. ∠s.
18. 1. 8 mi. 2. 80°.

PART II.

Exercise XXIX. (p. 101.)

1. 3·22 sq. in.
4. 8 sq. mi.; $\frac{1}{16}$ sq. in.
5. 2·75; 14·25; 22 sq. in.
6. 2·5 in.
7. 8·485 in. approx.
8. 0·8, 1·1, 0·78 sq. in.
9. 1·9, 2·95 sq. in.
10. 12·57 sq. in.
11. 6 sq. in.
12. 3,900 sq. yd.

Exercise XXX. (p. 107.)

1. 7·5.
2. 17·5.
3. 4·8.
4. 4.
5. 42.
6. 44.
7. 6.
8. 21.
9. 4·8.
10. 12.
11. 6·75.
12. 10·5.
13. 3·75.
14. 4·5; 4.
15. 4·8.
16. 15.
17. 4·8; 4·8.
18. 4·4.
19. 1000·
20. 6·2; 20.
21. 4 ft. to mile; $\frac{1}{2}$ in.
22. $\frac{1}{2}(xq + xr + yp + yq)$.
23. $\frac{1}{2}(pr + qr + qs)$.
24. $\frac{pq}{r}$.
25. 24; 12; 36.
26. $\frac{1}{2}(xy - ef)$.
27. 5; 10.
28. (i) 4; (ii) 5; (iii) 5·5; (iv) $\frac{1}{2}ac$; (v) $\frac{1}{2}(ad - bc)$
29. (i) 10; (ii) 11.
30. (i) 3·3; (ii) 6·4
31. (i) 14·7, 5·88; (ii) 57·2, 14·3.
32. 5·56.

Exercise XXXI. (p. 113.)

1. (i) 10; (ii) 50·0; (iii) 14·7; (iv) 6; (v) 48; (vi) 9·43; (vii) 45·1; (viii) 28; (ix) 21; (x) 18.
2. 15·0.
3. 5·75.
4. 4·57.
5. 30°.
6. 2·64.
7. $36\frac{3}{4}^\circ$.
8. 40°.
9. 4·07.
10. 5·80 or 10·6.
11. 29·1.

Exercise XXXII. (p. 115.)

6. 1·56 in.
14. 2·90 in.

Exercise XXXIII. (p. 120.)

1. 13, $\frac{25}{13}$.
2. 8, 3·6.
3. 5·66, $5\frac{4}{9}$.
4. 32·25.
5. $9\frac{3}{13}$.
6. 5·83.
7. 217.
8. 4·77.
10. 30.
11. 14,970.
12. 17·3; 1·975 ft.
13. 21·1.
14. 16·2 mi.
15. 60 yd.
16. 4·47.

17. 5. 18. 5. 19. 6·93. 20. 2·89.
21. 5. 22. 13. 25. $6\frac{2}{3}$. 26. $8\frac{1}{6}$.
27. $8\frac{5}{3}$ 28. 3·57. 29. 7. 30. 9·16.
31. 8·66. 32. 26·8. 33. 18. 34. 6·24.
35. Each side 60 sq. in.; 11·7 in. 36. 7·34.

Exercise XXXV. (p. 127.)

19. Part of the circle through B, A, D.

Exercise XXXVII. (p. 132.)

17. 20 in. 19. 3, 5.

Revision Papers 19-36. (pp. 134-140.)

19. 1. 120°. 3. 9 sq. in. **20.** 2. 36°. 4. 9·5 sq. in.
21. 1. 44°. **22.** 3. 20″.
23. 1. $\frac{1}{2}(x-y)$, $180-\frac{1}{2}(x-y)$. 3. 5·29.
24. 1. $2n-3$. 4. 4, 11·18. **25.** 1. 3·75.
26. 1. $\frac{1}{2}(xy+yz)$. 4. 20 sq. cm., 4·24 cm.
27. 1. 13·4 (5). 3. 2 in. **28.** 1. $2\frac{2}{3}$. 4. 5·5, 2·5; 17·3.
29. 1. $\frac{1}{2}[p(y+r)+q(r+s)+x(s-y)]$. 4. 13 in.
30. 1. 15, 9. 4. 9. **31.** 1. 300 sq. ft.
32. 1. 7·5 cm. 2. 2·16 sq. in. **33.** 1. 12; 5·66.
34. 1. 2 cm. 4. (ii) $\sqrt{(x^2-8x+416)}$; $x>6\frac{1}{9}$. **35.** 1. 2 in.

Exercise XXXVIII. (p. 146.)

1. 9·16. 2. 13. 3. 11·5. 4. $7\frac{1}{24}$.
5. 8·58, 0·58. 6. 5·38. 7. 3·46. 8. 5. 9. 4.
10. 8. 11. 3·12. 12. 5·22. 13. $\dfrac{x^2+y^2+z^2-2xz}{2(x-z)}$.

Exercise XXXIX. (p. 152.)

1. 40°. 2. 55°. 3. 110°. 4. 37°.
5. 107°. 6. 100°; 110°. 7. 54°; 99°. 8. 105°.
9. 72°. 10. 124°. 11. 54°. 12. 105°.

Exercise XLI. (p. 162.)

1. 30°, 45°, 105° or 15°, 30°, 135°.
2. $7\frac{1}{2}^\circ$, $22\frac{1}{2}^\circ$, 150° or $22\frac{1}{2}^\circ$, 30°, $127\frac{1}{2}^\circ$. 4. 3 : 1. 5. 46°, 37°.

Exercise XLII. (p. 167.)

1. 62°. **2.** 117°. **3.** 26°, 8°. **4.** 58°, 64°.
5. 103°, 90°, 77°, 90°. **6.** 94°, 8°. **7.** 120°.

Exercise XLIII. (p. 174.)

1. 3. **2.** 2·5, 1·5, 4·5. **3.** 8, 4, 3.
4. 5·3, 3·6, 4·5. **5.** 10·5, 1·5. **6.** 6. **7.** $1\frac{7}{8}$.
8. 32, 8. **9.** 3. **10.** 1·5, 2·5. **11.** ·5, 2·5.
12. 12. **13.** 19·1, 12. **14.** 7, 1. **15.** 4·45, 11·125.
16. $5-3\sqrt{2}=\cdot757$. **18.** $2\frac{1}{6}$.

Exercise XLV. (p. 181.)

1. 25·1 in., 50·3 sq. in.; 628 yd., 31,420 sq. yd. **2.** 0·8.
3. 1·1. **4.** 2·1. **5.** 5·89. **6.** 4·57.
7. 57° 18′. **8.** 3·2. **9.** 158·5. **11.** 84·8.
12. 21·5. **13.** 628; 408. **14.** $3\frac{5}{9}$. **15.** 25.
16. 314; 204. **17.** $\frac{5}{6}$. **18.** 288°. **19.** 48; 96.
20. 65·4; 78·5. **21.** 100,000,000 sq. m.; $\frac{1}{2}$. **22.** 8·2.
23. 9·21. **24.** 20·1. **25.** $2\frac{2}{3}$. **26.** 78·5.
27. 77·4. **28.** 828·5 sq. ft. **32.** ·0182 in. per sec.
33. 8; 14; $1\frac{5}{7}$. **34.** 6·86; 137; 186.

Exercise XLVI. (p. 187.)

5. 6·65. **17.** 0·64, 1·16, 1·93, 5·80. **18.** 1·46.
23. 3·11. **24.** 4·61.

Exercise XLVII. (p. 193.)

6. 4·47. **9.** 3·20. **14.** 2·66. **15.** 1·56.
16. 5·80. **17.** 1·32. **18.** 8·13. **23.** 5·60, 2·14.
24. 6·06, 4·02. **26.** 5·87, 2·23. **30.** 4·16.

Revision Papers 37-56. (pp. 195-201.)

37. 2. 55·2. 4. 17·14. **38.** 2. $3\frac{1}{4}$.
39. 2. 2·9. 3. 162°. **40.** 2. 11·2.
41. 2. 5, 7. 4. 51°. **42.** 4. 5·56 sq. in.
44. 3. 5·66, 8·485. **45.** 3. 47°.
46. 2. 60°, 80°. **47.** 2. $\frac{a^2+4h^2}{4h}$. 3. 55°, 40°.
48. 1. 13. **49.** 3. 15°.

50. 2. E. 25° N. **53.** 3. 17.

54. 1. On AB 10, on CD 20. 4. 43·2.

55. 1. 5. **58.** 4. $\frac{2V}{S}$.

Exercise L. (p. 212.)

1. (i), (ii), (iv). 2. 19. 3. $1\frac{3}{8}$; 2·67. 4. 5·85; 6·84.
5. 11; 1; 6·93. 6. 42·43. 7. 6·63.
8. 12·2. 10. Yes. 13. 3·5.
14. 5·45; 6·52; 7·97. 15. 9·17. 16. 10. 17. 12·7.

Exercise LI. (p. 218.)

1. 4, 10, 12. 2. 24 sq. cm., 4 cm. 3. 8.
4. 3·2. 5. 1. 6. 13, 6, $7\frac{1}{2}$. 7. 56 sq. cm.
8. $10\frac{2}{3}$, $8\frac{1}{3}$. 9. 20 ft. 10. 2·25. 11. 34 in.
12. $2\frac{1}{2}$ ft. 13. 6·5. 14. 3·54, 6·52. 15. 3, 30 mi.

Exercise LII. (p. 222.)

1. 6·325. 2. 5·57. 3. 8, 3. 4. 7·37, 1·63.
5. 1·97 8. 8·06. 9. 7·38, 3·38. 10. 7·24, 2·76.

PART III.

Exercise LIII. (p. 225.)

2. (iv) $1\frac{1}{4}$. 5. $1\frac{1}{4}$; 0 or 1. 7. 6. 8. $\frac{1}{18}$.
10. 3·2. 11. 6. 15. 2 : 5 ; 1 : 2. 16. 1·6 in.
18. 3·2. 21. $\dfrac{x \sim y}{2(x+y)}$. 22. $\dfrac{2xy}{x^2-y^2}$; $\dfrac{x-y}{x+y}$.
23. $\mathrm{AF} = \dfrac{x(p+q+r+s)}{q+r}$. 25. $\dfrac{1}{\lambda-1}$.

Exercise LIV. (p. 229.)

1. 3. 2. 5. 3. $\frac{1}{30}$ in. 4. 881 ; $\frac{1}{883}$ in. ; $\frac{640}{883}$, $\frac{243}{883}$ in.
5. $12\frac{1}{2}$. 6. $4\frac{1}{6}$. 7. 1·6. 8. 5·2. 9. 3 : 2. 23. 1.

Exercise LV. (p. 233.)

1. 7·5. 2. 7·2. 7. (i) 2·89 ; (ii) 10·3.

Exercise LVI. (p. 235.)

1. 3, 15. 2. 3·35. 4. 12. 5. $9\frac{3}{4}$. 6. 3 sq. in.

Exercise LVII. (p. 237.)

2. 8 or 4·5. 11. $2\frac{2}{3}$, 2 ; 6, 3 ; 8, $5\frac{1}{3}$.

Exercise LVIII. (p. 242.)

1. 120. 2. 4 ft. 4. $10^5 \times 8{\cdot}6$; $10^5 \times 2{\cdot}3$ mi.
5. 6′ 8″. 6. 66. 7. 14·4″. 8. 6·4, 7·2 cm.
9. $22\frac{1}{2}$. 10. 1·5, $3\frac{3}{4}$. 11. 5. 12. $8\frac{1}{3}$.
13. (i) $\frac{3}{4}$, $\frac{9}{8}$; (ii) $6\frac{1}{2}$; (iii) $2\frac{2}{3}$, $1\frac{1}{3}$; (iv) $5\frac{1}{3}$, $3\frac{5}{9}$. 15. 2·4.
16. 18, 8. 17. 7·2. 18. 14. 19. $3\frac{1}{2}$, 11.
20. 12·8, 5. 21. $8\frac{4}{7}$. 22. 4.
23. (i) $2\frac{1}{3}$; (ii) $7x+5y=35$. 24. 2·9. 25. 12.
26. $1\frac{5}{7}$. 27. 6. 11. 28. $3\frac{1}{8}$.
29. (i) 54′, 24′ ; (ii) 13″. 30. $3\frac{3}{11}$. 31. $y=\dfrac{fx}{u-f}$.
32. $y=\dfrac{fx}{u-f}$. 34. $y=\dfrac{fx}{u-f}$.

Exercise LIX. (p. 251.)

1. 6·325. 3. 6·08. 4. 7·36 or $-1·36$.
5. 5·29. 6. 3·29. 7. 5·00.

Exercise LX. (p. 254.)

2. 6. 3. $\frac{25}{13}$. 4. 10. 5. 2 or 10.
6. (i) 6; (ii) 12; (iii) 2·31; (iv) $21\frac{9}{11}$. 7. $4\frac{1}{2}$, $6\frac{1}{4}$.
8. $4\frac{1}{6}$. 11. 7·07; 13·04. 12. 0·707. 13. $\dfrac{p^2r}{q^2-p^2}$, $\dfrac{pqr}{q^2-p^2}$.

Exercise LXI. (p. 257.)

1. 12 sq. ft. 2. 40. 5. 9. 6. $101\frac{1}{4}$.
8. 4·2. 9. 16 : 4 : 3 : 9. 11. £$5\frac{1}{3}$.
12. $4\frac{17}{27}$. 14. 512. 15. 1·024. 16. 6.
17. 2*s*. 3*d*. 18. $9\frac{13}{27}$. 20. 40·5; 162.
28. $1, \frac{1}{2}, \pi, \dfrac{1}{4\pi}, \frac{1}{4}\sqrt{3}, \dfrac{3\sqrt{3}}{2}, 6, 4\pi$. 29. $1, \dfrac{\sqrt{3}}{9}, \dfrac{\pi}{6}, \dfrac{1}{6\pi^2}, \dfrac{\pi}{4}, \dfrac{\pi}{3}, \dfrac{\sqrt{2}}{12}$.

Exercise LXII. (p. 261.)

2. 3·63. 6. 2·27. 7. 2·68. 8. 5·36.

Exercise LXIII. (p. 263.)

7. 7·19, 2·19.

Exercise LXIV. (p. 264.)

2. 10. 3. 5·78. 4. 4·81. 6. 3·83.

Revision Papers 57-72. (pp. 265-271.)

57. 1. 8, 7. 2. QA = 60 ft. 2 in., PA = 25 ft. $5\frac{1}{2}$ in.
58. 3. $6\frac{1}{2}$, 9 in. **59.** 3. $\frac{3}{4}$.
61. 3. $9\frac{1}{3}$, $\frac{1}{2}$. **62.** 3. $6\frac{3}{4}$ in.
64. 3. 6, 10, 14 in. **65.** 2. 132.
66. 2. 4·47. 3. 24·4 in. **67.** 1. 6, 2·65. 3. 2·2.
68. 2. 0·69 or 23·3. 3. $5\frac{1}{2}$, $2\frac{3}{11}$. **69.** 3. 2.
70. 1. 3·2, 1·2, 4·4. **71.** 2. 68·7. 3. 2, $2\frac{2}{3}$.
72. 1. 4. 3. 4·8.

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